BÉZOUTIANS AND INJECTIVITY OF POLYNOMIAL MAPS

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Abstract. We prove that an endomorphism f of affine space is injective on rational points if its Bézoutian is constant. Similarly, f is injective at a given rational point if its reduced Bézoutian is constant. We also show that if the Jacobian determinant of f is invertible, then f is injective at a given rational point if and only if its reduced Bézoutian is constant.

1. Introduction

Let k be a field, and let $f = (f_1, \ldots, f_n)\colon \mathbb{A}^n_k \to \mathbb{A}^n_k$ be a polynomial morphism. In this note, we study the injectivity of f at the origin using the *multivariate Bézoutian.*

Definition 1.1. Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$. The *(multivariate) Bézoutian* of $f := (f_1(\mathbf{x}), \ldots, f_n(\mathbf{x}))$ is the determinant

$$\mathrm{Béz}(f) := \det(\Delta_{ij}) \in k[\mathbf{x}, \mathbf{y}],$$

where

$$\Delta_{ij} = \frac{f_i(y_1, \ldots, y_{j-1}, x_j, \ldots, x_n) - f_i(y_1, \ldots, y_j, x_{j+1}, \ldots, x_n)}{x_j - y_j}.$$

The *reduced Bézoutian* of f is $\overline{\mathrm{Béz}}(f) := \mathrm{Béz}(f) \bmod (f(\mathbf{x}), f(\mathbf{y}))$.

Definition 1.2. Let R be a polynomial ring over a field k. Let I be an ideal of R. If I is a proper ideal, then $k \subseteq R/I$. We say that an element $c \in R/I$ is *constant* if (i) I is a proper ideal and $c \in k$, or if (ii) I is not proper, in which case $c = 0$.

Multivariate Bézoutians generalize the classical Bézoutian of a univariate polynomial. They naturally arise in the study of global residues (see e.g. [SS75, BCRS96]). We will show that f is injective at a k-rational point q if $\overline{\mathrm{Béz}}(f - q)$ is constant.

Theorem 1.3. *Let k be a field, and let $f = (f_1, \ldots, f_n)\colon \mathbb{A}^n_k \to \mathbb{A}^n_k$ be a polynomial morphism with finite fibers. Let $q = (q_1, \ldots, q_n)$ be a k-rational point of $\mathbb{A}^n_k$. If $\overline{\mathrm{Béz}}(f - q)$ is constant, then $|f^{-1}(q)| \leq 1$.*

We will also see that f is injective on k-rational points if $\mathrm{Béz}(f)$ is constant.

Corollary 1.4. *If $\mathrm{Béz}(f)$ is constant, then f is injective on k-rational points.*

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In general, the constancy of $\mathrm{Béz}(f)$ (or $\overline{\mathrm{Béz}}(f-q)$) is a sufficient but not necessary condition for injectivity (see Example 4.11). In Lemma 4.5, we describe circumstances under which Theorem 1.3 gives a necessary and sufficient condition for injectivity at a rational point.

Bass, Connell, and Wright have shown that if k has characteristic 0 and $\mathrm{Jac}(f) \in k^{\times}$, then f is invertible if and only if f is injective on k-rational points [BCW82, Theorem 2.1]. In particular, Theorem 1.3 and Lemma 4.5 give a reformulation of the Jacobian conjecture in characteristic 0.

Corollary 1.5. *Let k be an algebraically closed field of characteristic 0, and let $f = (f_1, \ldots, f_n)\colon \mathbb{A}^n_k \to \mathbb{A}^n_k$ be a polynomial morphism. Assume $\mathrm{Jac}(f) \in k^{\times}$. Then $\overline{\mathrm{Béz}}(f-q)$ is constant for all $q \in \mathbb{A}^n_k(k)$ if and only if f is invertible.*

Note that $\mathrm{Béz}(f) = \mathrm{Béz}(f-q)$ for any k-rational point q. In particular, if $\mathrm{Béz}(f)$ is constant, then $\overline{\mathrm{Béz}}(f-q)$ is constant for all $q \in \mathbb{A}^n_k(k)$. This gives a sufficient but not necessary criterion for the Jacobian conjecture.

Corollary 1.6. *Let k be a field of characteristic 0, and let $f = (f_1, \ldots, f_n)\colon \mathbb{A}^n_k \to \mathbb{A}^n_k$ be a polynomial morphism. Assume $\mathrm{Jac}(f) \in k^{\times}$. If $\mathrm{Béz}(f)$ is constant, then f is invertible.*

The key observation leading to Theorem 1.3 is that $\overline{\mathrm{Béz}}(f-q)$ records information about the dimension of $k[\mathbf{x}]/(f-q)$ as a k-vector space. We will recall the relevant details about Bézoutians in Section 2. We will then prove Theorem 1.3 in Section 3. Finally, we discuss Theorem 1.3 in the context of the Jacobian conjecture in Section 4.

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2. Bézoutians

Throughout this section, let $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ be a morphism with finite fibers. This ensures that $(f_1, \ldots, f_n)$ is a complete intersection ideal, which allows us to utilize the multivariate Bézoutian [BCRS96, Section 3].

Remark 2.1. As noted by Scheja–Storch [SS75, p. 182] and Becker–Cardinal–Roy–Szafraniec [BCRS96], the Bézoutian records information about the dimension of $k[\mathbf{x}]/(f)$ as a k-vector space. To see this, consider the isomorphism

$$\mu\colon \frac{k[\mathbf{x}]}{(f)} \otimes_k \frac{k[\mathbf{x}]}{(f)} \to \frac{k[\mathbf{x},\mathbf{y}]}{(f(\mathbf{x}), f(\mathbf{y}))}$$

defined by $\mu(a(\mathbf{x}) \otimes b(\mathbf{x})) = a(\mathbf{x})b(\mathbf{y})$. The inverse is characterized by $\mu^{-1}(x_i) = x_i \otimes 1$ and $\mu^{-1}(y_i) = 1 \otimes x_i$. Since μ is an isomorphism, there is an element $B \in k[\mathbf{x}]/(f) \otimes_k k[\mathbf{x}]/(f)$

such that $\mu(B) = \overline{\text{Béz}}(f)$. Moreover, given a basis $\{c_i\}$ for $k[\mathbf{x}]/(f)$, there exists a basis $\{d_i\}$ for $k[\mathbf{x}]/(f)$ such that $B = \sum_i c_i \otimes d_i$ [BCRS96, Theorem 2.10(iii)].

Proposition 2.2. *If* $\overline{\text{Béz}}(f-q)$ *is constant, then* $\dim_k k[\mathbf{x}]/(f-q) \leq 1$.

Proof. We will prove that if $\overline{\text{Béz}}(f)$ is constant, then $\dim_k k[\mathbf{x}]/(f) \leq 1$. The same proof holds after replacing f with $f-q$. Given any basis $\{c_1, \ldots, c_m\}$ of $k[\mathbf{x}]/(f)$, write $\overline{\text{Béz}}(f) = \sum_{i,j} B_{ij} c_i(\mathbf{x}) c_j(\mathbf{y})$, where $B_{ij} \in k$. The $m \times m$ matrix (B_{ij}) is non-singular by [BMP21b, Theorem 1.2], so (B_{ij}) must contain at least m non-zero entries. In particular, the number of non-zero terms of $\sum_{i,j} B_{ij} c_i(\mathbf{x}) c_j(\mathbf{y})$ is at least $m = \dim_k k[\mathbf{x}]/(f)$.

First, suppose $\overline{\text{Béz}}(f) \in k^\times$. Pick a monomial basis $\{c_1, \ldots, c_m\}$ of $k[\mathbf{x}]/(f)$, so that $\sum_{i,j} B_{ij} c_i(\mathbf{x}) c_j(\mathbf{y})$ consists of a single non-zero term, so $1 \geq \dim_k k[\mathbf{x}]/(f)$ (and in fact, equality holds). Next, if $\overline{\text{Béz}}(f) = 0$, then $m = \dim_k k[\mathbf{x}]/(f) = 0$. $\square$

Let $\text{Jac}(f) := \det(\frac{\partial f_i}{\partial x_j})$ be the Jacobian of f, and let $\delta \colon k[\mathbf{x}, \mathbf{y}] \to k[\mathbf{x}]$ be given by $\delta(a(\mathbf{x}, \mathbf{y})) = a(\mathbf{x}, \mathbf{x})$. We can recover $\text{Jac}(f)$ from $\text{Béz}(f)$. This appears in [SS75, p. 184] and, modulo (f), in [BCRS96, p. 90], but we recall the details here.

Proposition 2.3. *We have* $\delta(\text{Béz}(f)) = \text{Jac}(f)$.

Proof. Note that δ is a ring homomorphism, so it suffices to show that $\delta(\Delta_{ij}) = \frac{\partial f_i}{\partial x_j}$. The result follows by taking a formal partial derivative, as we now explain. Let

$$f_{ij}(\mathbf{x}, y_j) = \frac{f_i(\mathbf{x}) - f_i(x_1, \ldots, x_{j-1}, y_j, x_{j+1}, \ldots, x_n)}{x_j - y_j},$$

so that $\delta(\Delta_{ij}) = f_{ij}(\mathbf{x}, x_j)$. Since Δ_{ij} is a polynomial, $\delta(\Delta_{ij})$ and f_{ij} are polynomials as well. Now

$$\begin{aligned}
\frac{\partial f_i}{\partial x_j} &= \frac{\partial}{\partial x_j} \left(f_i(\mathbf{x}) - f(x_1, \ldots, x_{j-1}, y_j, x_{j+1}, \ldots, x_n) \right) \\
&= \frac{\partial}{\partial x_j} \left(f_{ij}(\mathbf{x}, y_j) \cdot (x_j - y_j) \right) \\
&= \frac{\partial f_{ij}}{\partial x_j} \cdot (x_j - y_j) + f_{ij}(\mathbf{x}, y_j).
\end{aligned}$$

Thus

$$\begin{aligned}
\delta\left(\frac{\partial f_i}{\partial x_j}\right) &= \delta\left(\frac{\partial f_{ij}}{\partial x_j} \cdot (x_j - y_j) + f_{ij}(\mathbf{x}, y_j)\right) \\
&= 0 + f_{ij}(\mathbf{x}, x_j) \\
&= \delta(\Delta_{ij}).
\end{aligned}$$

Since $\frac{\partial f_i}{\partial x_j} \in k[\mathbf{x}]$, we have $\delta(\frac{\partial f_i}{\partial x_j}) = \frac{\partial f_i}{\partial x_j}$, which proves the desired result. $\square$

Remark 2.4. If $\mathrm{Jac}(f) \in k^\times$ and $\overline{\mathrm{Béz}}(f)$ is constant, then Proposition 2.3 implies that $\overline{\mathrm{Béz}}(f) \neq 0$.

Example 2.5. Let $f = (x_1^2, x_2^2, x_3^2)$. The set $\{1, x_1, x_2, x_3, x_1x_2, x_1x_3, x_2x_3, x_1x_2x_3\}$ is a basis for $k[\mathbf{x}]/(f)$. Let

$$\begin{aligned} B = x_1x_2x_3 \otimes 1 + x_2x_3 \otimes x_1 + x_1x_3 \otimes x_2 + x_1x_2 \otimes x_3 \\ + x_1 \otimes x_2x_3 + x_2 \otimes x_1x_3 + x_3 \otimes x_1x_2 + 1 \otimes x_1x_2x_3. \end{aligned}$$

By Definition 1.1, we have

$$\begin{aligned} \mathrm{Béz}(f) = x_1x_2x_3 + x_2x_3y_1 + x_1x_3y_2 + x_1x_2y_3 \\ + x_1y_2y_3 + x_2y_1y_3 + x_3y_1y_2 + y_1y_2y_3. \end{aligned}$$

One can readily check that $\mu(B) = \mathrm{Béz}(f)$. Moreover, $\delta(\mathrm{Béz}(f)) = 8x_1x_2x_3$, which is equal to $\mathrm{Jac}(f)$ (see Proposition 2.3).

3. Proof of Theorem 1.3

Let $q = (q_1, \ldots, q_n) \in \mathbb{A}_k^n$ be a k-rational point. As a consequence of the structure theorem for Artinian rings, the dimension of $k[\mathbf{x}]/(f-q)$ as a k-vector space is closely related to the fiber cardinality $|f^{-1}(q)|$.

Proposition 3.1. *Let $(f-q) = (f_1 - q_1, \ldots, f_n - q_n)$ be an ideal in $k[\mathbf{x}]$. Suppose that $f^{-1}(q) = \{p_1, \ldots, p_m\}$ is a finite set of points. Then $\dim_k k[\mathbf{x}]/(f-q) \geq |f^{-1}(q)|$.*

Proof. Since $f^{-1}(q) = \mathbb{V}(f-q)$ is a finite set, $k[\mathbf{x}]/(f-q)$ is Artinian by [Sta21, Lemma 00KH]. Let $\mathfrak{m}_i$ be the maximal ideal in $k[\mathbf{x}]$ corresponding to p_i. By the structure theorem for Artinian rings (see [Sta21, Lemma 00JA] or [AM69, Theorem 8.7]), there is an isomorphism

$$\frac{k[\mathbf{x}]}{(f-q)} \cong \prod_{i=1}^m \frac{k[\mathbf{x}]_{\mathfrak{m}_i}}{(f-q)}.$$

Thus $\dim_k k[\mathbf{x}]/(f-q) = \sum_{i=1}^m \dim_k k[\mathbf{x}]_{\mathfrak{m}_i}/(f-q)$, which implies the claim. □

We are now prepared to prove Theorem 1.3 and Corollary 1.4.

Proof of Theorem 1.3. By Proposition 2.2, we have $\dim_k k[\mathbf{x}]/(f-q) \leq 1$, so Proposition 3.1 implies that $|f^{-1}(q)| \leq 1$. □

Proof of Corollary 1.4. Note that for any k-rational point q, we have $\mathrm{Béz}(f-q) = \mathrm{Béz}(f)$. Thus if $\mathrm{Béz}(f) \in k$, then $\overline{\mathrm{Béz}}(f-q) \in k$ for all $q \in \mathbb{A}_k^n(k)$. By Theorem 1.3, f is injective on k-rational points. □

4. Drużkowski morphisms with constant Bézoutian

If we assume that $\mathrm{Jac}(f) \in k$, then we get slightly stronger injectivity results. By the work of Bass, Connell, and Wright [BCW82, Theorem 2.1], we can study the Jacobian conjecture by studying the injectivity of morphisms with $\mathrm{Jac}(f) \in k^\times$. We start with the following standard result.

Proposition 4.1. *If $f\colon \mathbb{A}^n_k \to \mathbb{A}^n_k$ has $\mathrm{Jac}(f) \in k^\times$, then f has finite fibers.*

Proof. Let $X = k[\mathbf{x}]/(f)$, and recall that the module of Kähler differentials $\Omega_{X/k}$ is the cokernel of the Jacobian matrix $(\frac{\partial f_i}{\partial x_j})$. Since $\mathrm{Jac}(f) \in k^\times$, we have that $\Omega_{X/k} = 0$ and hence f is unramified. By [Sta21, Lemma 02V5], f is locally quasi-finite. Since $\mathbb{A}^n_k$ is Noetherian, $f\colon \mathbb{A}^n_k \to \mathbb{A}^n_k$ is quasi-compact. Thus [Sta21, Lemma 01TJ] implies that f is quasi-finite. In particular, f has finite fibers [Sta21, Lemma 02NH]. □

Remark 4.2. The statement that f has finite fibers is equivalent to $\mathbb{V}(f-q)$ being a finite set for all q. Assuming $\mathrm{Jac}(f) \in k^\times$, it was shown by van den Essen [vdE00, Theorem 1.1.32] that $|\mathbb{V}(f-q)| \leq [k(\mathbf{x}) : k(\mathbf{f})]$. Miranda-Neto also proved the finiteness of $\mathbb{V}(f-q)$ using derivations and differentials [MN19, Theorem 3.1].

We saw in Proposition 3.1 that $\dim_k k[\mathbf{x}]/(f-q) \geq |f^{-1}(q)|$. Assuming that k is algebraically closed of characteristic 0 and $\mathrm{Jac}(f) \in k^\times$, this inequality is an equality.

Proposition 4.3. *Let k be an algebraically closed field of characteristic 0. If $f\colon \mathbb{A}^n_k \to \mathbb{A}^n_k$ has $\mathrm{Jac}(f) \in k^\times$, then $\dim_k k[\mathbf{x}]/(f-q) = |f^{-1}(q)|$. (See also [MN19, Corollary 3.2].)*

Proof. We need to show that if $p \in f^{-1}(q)$ with corresponding maximal ideal $\mathfrak{m}$, then $\dim_k k[\mathbf{x}]_\mathfrak{m}/(f-q) = 1$. Since f has finite fibers, $k[\mathbf{x}]_\mathfrak{m}/(f-q)$ is a local Artin ring. In particular, $k[\mathbf{x}]_\mathfrak{m}/(f-q)$ is a finitely generated algebra over its residue field. Moreover, the residue field is k, since k is algebraically closed and $\mathfrak{m}$ is a closed point. We will conclude by showing that $k[\mathbf{x}]_\mathfrak{m}/(f-q)$ is in fact a field and noting that any finitely generated k-algebra is isomorphic to k (since k is algebraically closed).

Since k has characteristic 0, [SS75, (4.7) Korollar] implies that $\mathrm{Jac}(f)$ generates the socle of $k[\mathbf{x}]_\mathfrak{m}/(f-q)$, which is the annihilator of the maximal ideal $\mathfrak{m}$. That is, the maximal ideal of $k[\mathbf{x}]_\mathfrak{m}/(f-q)$ is annihilated by a scalar, so this maximal ideal must be the zero ideal. In particular, $k[\mathbf{x}]_\mathfrak{m}/(f-q)$ is a field.

Alternatively, one can note that $\mathrm{Jac}(f) \in k^\times$ implies that $\mathbb{V}(f-q)$ is smooth as an affine scheme. Moreover, $\mathbb{V}(f-q)$ has Krull dimension zero by Proposition 4.1. Since $\operatorname{char} k = 0$, it follows that $\mathbb{V}(f-q)$ is regular, so $k[\mathbf{x}]_\mathfrak{m}/(f-q)$ is a regular local ring over an algebraically closed field. By the Cohen structure theorem, the $\mathfrak{m}$-adic completion of $k[\mathbf{x}]_\mathfrak{m}/(f-q)$ is a ring of power series over k in 0 generators (i.e. k itself), so $k[\mathbf{x}]_\mathfrak{m}/(f-q) \cong k$ as rings. But this suffices to prove that $\dim_k k[\mathbf{x}]_\mathfrak{m}/(f-q) = 1$. □

Remark 4.4. In the course of Proposition 4.3, we have shown that if k is a field of characteristic 0 and $f\colon \mathbb{A}^n_k \to \mathbb{A}^n_k$ has $\mathrm{Jac}(f) \in k^\times$, then the ideal $(f_1 - q_1, \ldots, f_n - q_n) \subset$

$k[\mathbf{x}]$ is radical. Indeed, the structure theorem for Artinian rings allows us to decompose $k[\mathbf{x}]/(f-q)$ as a product of local Artinian rings, each of which is a field by [SS75, (4.7) Korollar]. In particular, $(f-q)$ is a finite intersection of maximal ideals and is hence radical. This gives a proof of [MN19, Theorem 3.1] not relying on derivations or differentials, as asked by Miranda-Neto [MN19, Remark 3.3].

If $\operatorname{Jac}(f) \in k^\times$, we get a converse to Theorem 1.3 (assuming k is algebraically closed with $\operatorname{char} k = 0$).

Lemma 4.5. *Let k be an algebraically closed field of characteristic 0. If $\operatorname{Jac}(f) \in k^\times$ and f is injective at $q \in \mathbb{A}^n_k(k)$, then $\overline{\text{Béz}}(f-q)$ is constant.*

Proof. Since f is injective at q, Proposition 4.3 implies that $\dim_k k[\mathbf{x}]/(f-q) \leq 1$. If $\dim_k k[\mathbf{x}]/(f-q) = 0$, then $\overline{\text{Béz}}(f-q) = 0 \in k$. If $\dim_k k[\mathbf{x}]/(f-q) = 1$, then $k[\mathbf{x},\mathbf{y}]/(f(\mathbf{x})-q, f(\mathbf{y})-q) \cong k \otimes_k k \cong k$. Thus $\overline{\text{Béz}}(f-q) \in k$, as desired. □

Corollary 1.5 follows from Theorem 1.3, Lemma 4.5, and [BCW82, Theorem 2.1]. Using Corollary 1.6, we can prove the Jacobian conjecture for any morphism whose Bézoutian is a constant. An important class of morphisms to consider are Drużkowski morphisms.

Definition 4.6. A morphism $f\colon \mathbb{A}^n_k \to \mathbb{A}^n_k$ is called a *Drużkowski morphism* if f is of the form $(x_1 + (\sum_{i=1}^n a_{1i}x_i)^3, \ldots, x_n + (\sum_{i=1}^n a_{ni}x_i)^3)$ with $\operatorname{Jac}(f) \in k^\times$. We also say that f is the Drużkowski morphism *determined by* the matrix (a_{ij}).

It was proved by Drużkowski [Dru83, Theorem 3] that if the Jacobian conjecture is true for all Drużkowski morphisms over a field k of characteristic 0, then the Jacobian conjecture is true over k. By [BCW82, (1.1) Remark 4], the Jacobian conjecture over $\mathbb{C}$ implies the Jacobian conjecture over all fields of characteristic 0.

If (a_{ij}) is strictly upper triangular or strictly lower triangular, then the Drużkowski morphism determined by (a_{ij}) has constant Bézoutian. This allows us to recover the well-known solution of the Jacobian conjecture for such morphisms [Tru15, Theorem 1.8]:

Proposition 4.7. *Let k be an algebraically closed field of characteristic 0. Suppose $a_{ij} = 0$ either for all $i \geq j$ or for all $i \leq j$. Then the morphism*

$$f := (x_1 + (\textstyle\sum_{j=1}^n a_{1j}x_j)^3, \ldots, x_n + (\sum_{j=1}^n a_{nj}x_j)^3)\colon \mathbb{A}^n_k \to \mathbb{A}^n_k$$

is invertible and has $\operatorname{Jac}(f) = 1$.

Proof. First suppose $a_{ij} = 0$ for all $i \geq j$. Since $a_{ij} = 0$ for $i > j$, we have $\Delta_{ij} = 0$ for $i > j$. Since $a_{ii} = 0$ for all i, we have $\Delta_{ii} = 1$ for all i. Thus $\text{Béz}(f) = \operatorname{Jac}(f) = 1$, and f is invertible by Corollary 1.6. Symmetrically, if $a_{ij} = 0$ for all $i \leq j$, then we again have $\text{Béz}(f) = \operatorname{Jac}(f) = 1$. □

Proposition 4.7 follows from [Dru83, Theorem 5] when the rank of (a_{ij}) is 0, 1, 2, or $n-1$. As mentioned in [Dru83, Remark 6], f is a Drużkowski morphism (in particular, $\mathrm{Jac}(f) \in k^\times$) only if $\mathrm{rank}(a_{ij}) < n$. More strongly, Drużkowski proved that if the Jacobian conjecture is true for all Drużkowski morphisms with $(a_{ij})^2 = 0$, then the Jacobian conjecture is true in general [Dru01, Theorem 2].

Since every nilpotent matrix is similar to a strictly upper triangular matrix, and since an invertible matrix P determines an automorphism $P\colon \mathbb{A}_k^n \to \mathbb{A}_k^n$ given by $P(\mathbf{x}) = P\mathbf{x}^T$, Proposition 4.7 gives a potential to approach the Jacobian conjecture [Mei95]. Given an invertible matrix P and a Drużkowski morphism f determined by (a_{ij}), we would like to find automorphisms $S, T\colon \mathbb{A}_k^n \to \mathbb{A}_k^n$ such that $S \circ f \circ T$ is the Drużkowski morphism determined by $P(a_{ij})P^{-1}$. It suffices to reduce to the case where P is an elementary matrix. If P is a permutation matrix, then finding S, T is straightforward.

Proposition 4.8. *[GTGZ99, Proposition 3.1] Let P be a permutation matrix. If f is the Drużkowski morphism determined by (a_{ij}), then $P \circ f \circ P^{-1}$ is the Drużkowski morphism determined by $P(a_{ij})P^{-1}$.*

Paired with Proposition 4.7, we recover [GTGZ99, Theorem 3.2]. However, Meisters proved that not all nilpotent matrices are cubic similar to a strictly upper triangular matrix [Mei95], which suggests that finding such automorphisms S, T is not trivial. Indeed, the obvious trick does not quite work when P is a row multiplication matrix, as we show in Remark 4.9. Row addition matrices seem to be even more problematic than row multiplication matrices.

Remark 4.9. Let $\mathbf{e}_i$ be the i^{th} standard column vector. Let $D_i = (\mathbf{e}_1 \ \cdots \ m\mathbf{e}_i \ \cdots \ \mathbf{e}_n)$ for some $m \in k^\times$. If f is the Drużkowski morphism determined by (a_{ij}), then $D_i^3 \circ f \circ D_i^{-1}$ has matrix $D_i(a_{ij})D_i^{-1}$ but is not quite a Drużkowski morphism. Indeed, we compute

$$\begin{aligned} D_i^3 \circ f \circ D_i^{-1} = (&x_1 + (m^{-1}a_{1i}x_i + \textstyle\sum_{j \neq i} a_{1j}x_j)^3, \ldots, \\ &m^2 x_i + (a_{ii}x_i + \textstyle\sum_{j \neq i} m a_{ij}x_j)^3, \ldots, \\ &x_n + (m^{-1}a_{ni}x_i + \textstyle\sum_{j \neq i} a_{nj}x_j)^3). \end{aligned}$$

This is a Drużkowski morphism if and only if $m = \pm 1$, since we need $m^2 x_i = x_i$.

By Propositions 4.7 and 4.8 and Remark 4.9, we get the following corollary.

Corollary 4.10. *If (a_{ij}) is conjugate to a strictly upper triangular matrix via permutation matrices and ± 1 row multiplications, then the Drużkowski morphism determined by (a_{ij}) is invertible.*

Example 4.11. If all entries of (a_{ij}) are non-negative real numbers, then (a_{ij}) is nilpotent if and only if it is permutation-similar to a strictly upper triangular matrix. Unfortunately, Corollary 4.10 does not cover all Drużkowski morphisms. For example, the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

is not conjugate to a strictly upper triangular matrix via permutation matrices and ± 1 row multiplications [GTGZ99, (1.6)]. Moreover, the Drużkowski morphism f_A determined by A has $\text{Béz}(f_A) \notin k$, so one cannot hope for Corollary 1.6 to be of use for general Drużkowski morphisms. Nevertheless, the Jacobian conjecture is true for f_A, so $\overline{\text{Béz}}(f_A - q) = 1$ for all q.

It is known that Drużkowski morphisms are injective at the origin [Dru83, Proposition 1], so $\overline{\text{Béz}}(f) = 1$ for any Drużkowski morphism f by Lemma 4.5. Indeed, $\text{Jac}(f) = 1$ for any Drużkowski morphism, so $1 \equiv \text{Jac}(f) \mod (f)$. Since $\overline{\text{Béz}}(f) \in k$ and δ is injective on k (see Section 2), we have $\overline{\text{Béz}}(f) = 1$. Similarly, $\text{Jac}(f - q) = 1$ for any k-rational point $q \in \mathbb{A}^n_k$, so if $\overline{\text{Béz}}(f - q)$ is constant, then $\overline{\text{Béz}}(f - q) = 1$.

Question 4.12. *Given a Drużkowski morphism f, is $\overline{\text{Béz}}(f - q) = 1$ for all $q \in \mathbb{A}^n_k(k)$?*

Since $\overline{\text{Béz}}(f) = 1$, we have $\text{Béz}(f) \equiv 1 \mod (f(\mathbf{x}), f(\mathbf{y}))$. Since $\text{Béz}(f) = \text{Béz}(f - q)$ for any q, Question 4.12 is asking if $\text{Béz}(f) \equiv 1 \mod (f(\mathbf{x}) - q, f(\mathbf{y}) - q)$ for all q.

We include a basic Sage script [McK21] that takes as input a matrix A and a k-rational point $q \in \mathbb{A}^n_k$ and returns as output $\text{Jac}(f_A - q)$ and $\overline{\text{Béz}}(f_A - q)$ of the corresponding Drużkowski morphism f_A. This script is based off of code written jointly with Thomas Brazelton and Sabrina Pauli for computing the $\mathbb{A}^1$-degree via the Bézoutian [BMP21a].

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