BOUNDING THE SIGNED COUNT OF REAL BITANGENTS TO PLANE QUARTICS

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Abstract. Using methods from enriched enumerative geometry, Larson and Vogt gave a signed count of the number of real bitangents to real smooth plane quartics. This signed count depends on a choice of a distinguished line. Larson and Vogt proved that this signed count is bounded below by 0, and they conjectured that the signed count is bounded above by 8. We prove this conjecture using real algebraic geometry, plane geometry, and some properties of convex sets.

1. Introduction

In this note, we prove an upper bound to the signed count of real bitangents to real smooth plane quartics introduced by Larson and Vogt [LV21]. Using tools from $\mathbb{A}^1$-enumerative geometry (also called *quadratic* or *enriched* enumerative geometry), Larson and Vogt define the sign $\mathrm{QType}_L(B)$ (relative to an auxiliary line $L \subset \mathbb{P}^2$) of a bitangent B to a real smooth plane quartic Q. They then define the *signed count*

$$s_L(Q) := \sum_{B \text{ real bitangent}} \mathrm{QType}_L(B).$$

If $L \cap Q(\mathbb{R}) = \varnothing$, then $s_L(Q) = 4$ [LV21, Theorem 1]. In general, $s_L(Q)$ is a non-negative even integer [LV21, Proposition 4.3]. Larson and Vogt give examples of lines and quartics such that $s_L(Q) = 0, 2, 4, 6,$ and 8. This leads to the conjecture that $s_L(Q) \in \{0, 2, 4, 6, 8\}$ for any choice of line and quartic [LV21, Conjecture 2]. Using tools from tropical geometry, Markwig, Payne, and Shaw give a tropical criterion for quartics that satisfy $s_L(Q) \in \{0, 2, 4\}$ [MPS22, Theorem 5.2].

We gather a few ideas from real algebraic geometry and plane geometry to prove $s_L(Q) \leq 8$ for any choice of plane quartic and auxiliary line. Paired with Larson and Vogt's lower bound and examples, it follows that $s_L(Q) \in \{0, 2, 4, 6, 8\}$.

Theorem 1.1. *Let $Q \subset \mathbb{P}^2$ be a real smooth plane quartic. For any admissible line L, the signed count $s_L(Q)$ is at most 8. In particular, $s_L(Q) \in \{0, 2, 4, 6, 8\}$.*

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2. A few results on plane geometry and convex sets

In this section, we collect a few results on plane geometry and convex sets that we will need to prove Theorem 1.1.

Lemma 2.1. *Let $K \subset \mathbb{R}^n$ a compact set and $L \subset \mathbb{R}^n$ a line. If $|L \cap K| \geq 2$, then $|L \cap \partial K| \geq 2$.*

Proof. We identify L with the real line $\mathbb{R}$. Then $L \cap K$ is a compact subset of $\mathbb{R}$ with at least two elements. In particular, its minimum and maximum are different points on the boundary of $L \cap K$. Thus L intersects the boundary of K in at least two points. □

Corollary 2.2. *Let $K \subset \mathbb{R}^n$ a compact set and $L \subset \mathbb{R}^n$ a line. If $L \cap K^\circ \neq \varnothing$, then $|L \cap \partial K| \geq 2$.*

Proof. The condition $L \cap K^\circ \neq \varnothing$ implies that $|L \cap K| = \infty$, so we can apply the previous lemma. □

Corollary 2.3. *Let $I \subset \mathbb{R}^2$ be a line segment with endpoints a and b. Let $C \subset \mathbb{R}^2$ be a Jordan arc with endpoints a and b. Assume that $|I \cap C|$ is finite. Then any line $L \subset \mathbb{R}^2$ that meets I also meets C.*

Proof. Let $\varphi\colon [s,t] \to \mathbb{R}^2$ an injective continuous map with $\varphi(t_0) = a$ and $\varphi(t_1) = b$ whose image is C. Let $s = t_0 < \cdots < t_m = t$ such that $\varphi^{-1}(I \cap C) = \{t_0, \ldots, t_m\}$. Choose $0 \leq i < m$ such that L intersects the line segment I' with end points $\varphi(t_i)$ and $\varphi(t_{i+1})$. By replacing I by I' and C by $\varphi([t_i, t_{i+1}])$ we can restrict to the case that $I \cap C$ consists of the endpoints of I only. This means that $I \cup C$ is a Jordan curve. Without loss of generality we may assume L intersects I transversely in exactly one point and that this point is not on C. This implies that L intersects the interior (in the sense of the Jordan curve theorem) of the Jordan curve $I \cup C$ whose compact closure we denote by K. By Corollary 2.2 L intersects $\partial K = I \cup C$ in at least two points. Thus $L \cap C \neq \varnothing$ because $|I \cap L| = 1$. □

Lemma 2.4. *Let $K \subset \mathbb{R}^n$ a compact convex set and $L \subset \mathbb{R}^n$ a line. If $L \cap K^\circ \neq \varnothing$, then $|L \cap \partial K| = 2$.*

Proof. By assumption $L \cap K$ is a line segment spanned by two different boundary points v and w of K and there is some u on $L \cap K^\circ$. Let $u \in U \subset K$ an open neighbourhood of u in K. It remains to show that every point u' on $L \cap K$ other than v, w is in the interior of K. Without loss of generality, we can assume that u' is in the line segment spanned by v and u. Thus $u' = \mu v + (1-\mu)u$ for some $0 \leq \mu < 1$. The set

$$U' = \{\mu v + (1-\mu)x \mid x \in U\}$$

is then an open neighbourhood of u' in K. □

Lemma 2.5. *Let $Q \subset \mathbb{R}^2$ a convex quadrilateral. A line that intersects Q intersects at least two of its edges.*

Proof. Let L be a line that intersects Q. If L contains a vertex of Q, then it intersects the two edges that contain this vertex. Thus assume that L does not contain a vertex of Q. This implies that L intersects the interior of Q and that it intersects each edge in at most one point. By Lemma 2.4 it intersects the boundary of Q, which is the union of all edges, in two points. This implies the claim. $\square$

Lemma 2.6. *Let $Q \subset \mathbb{R}^2$ a convex quadrilateral. Let L be a line that intersects both diagonals of Q. Then L intersects two opposite edges of Q.*

Proof. First consider the case that L does not intersect the interior of Q. Then L intersects Q either in a single vertex or in an entire edge. In the former case L intersects only one diagonal of Q. In the latter case L intersects opposite edges.

Now assume that L intersect the interior of Q. Then by Lemma 2.4 the line L intersects the boundary of Q in exactly two different points P_1 and P_2. First assume that P_1 is a vertex of Q. Then P_2 cannot lie on an edge E that contains P_1 because then L intersects the interior of Q. Thus L intersects Q in at least three edges and therefore in particular in two opposite edges.

Finally assume that P_1 and P_2 both are not a vertex of Q. Then P_1 and P_2 lie on two different edges E_1 and E_2. Assume for the sake of a contradiction that E_1 and E_2 are not opposite. Then the convex hull of $E_1 \cup E_2$ is a triangle T with edges E_1, E_2 and D where D is a diagonal of Q. Because L does not intersect any vertex of T, it intersects the interior of T. Therefore, the two points P_1 and P_2 are the only intersection points of L with the boundary of T by Lemma 2.4. In particular, our line L does not intersect D. $\square$

Lemma 2.7. *Let $Q \subset \mathbb{R}^2$ be a convex quadrilateral. Let $E, E' \subset \partial Q$ be two opposite edges of Q. Let x be the intersection of the diagonals of Q. Then the set of points that lie on some line passing through E and E' is equal to the union of Q and the set of points that lie on some line passing through E and x (see Figure 1).*

Proof. Since Q is the convex hull of its four vertices, it is also the convex hull of E and E'. Thus the latter set is contained in the former and it remains to show that if a point $y \in \mathbb{R}^2 \backslash Q$ lies on a line L through E and E', then y lies on a line through E and x.

Let L' be the line through x and y. We will show that $L' \cap E$ is nonempty. By assumption, L' passes through x and hence both diagonals of Q. Thus if $L' \cap E = \varnothing$, then Lemma 2.6 implies that L' passes through the other pair of opposite edges of Q. Thus L and L' meet complementary pairs of opposite edges of Q, so L and L' must intersect within Q. But $y \in L \cap L'$ lies outside Q by assumption. We thus either contradict this assumption or find that $|L \cap L'| \geq 2$, another contradiction. $\square$

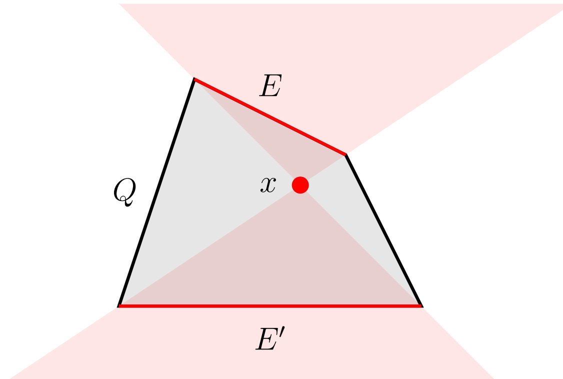

FIGURE 1. Support of lines through opposite edges of a quadrilateral

3. Proving the conjecture

Let $Q \subset \mathbb{P}^2$ a real smooth plane quartic curve. We say that a real line $L \subset \mathbb{P}^2$ is *admissible* if it is disjoint from $Q \cap B$ for every real bitangent B of Q. The QType [LV21, Definition 1.2] of a real bitangent B is the sign by which B should be counted.

Definition 3.1. Let $f \in \mathbb{R}[x_0, x_1, x_2]$ be a homogeneous degree 4 polynomial such that $\mathbb{V}(f) = Q$. Let B be a bitangent to Q, and denote $Q \cap B = 2Z$ for $Z = z_1 + z_2$ a degree 2 divisor. Let L be an admissible real line. Denote by ∂_L a derivation with respect to a linear form vanishing along L. The *QType of B with respect to L* is defined to be

$$\mathrm{QType}_L(B) := \mathrm{sign}(\partial_L f(z_1) \cdot \partial_L f(z_2)) \in \{+1, -1\}.$$

By [LV21, Equ. (6)] a real bitagent can have negative QType only if it intersects Q in real points only. In this case we say that the bitangent is *split*. Geometrically, a split bitangent B has QType $+1$ if all connected components of $Q(\mathbb{R})$ that it meets lie in the same component of $\mathbb{R}^2 \backslash B$ and QType -1 if it separates the connected components of $Q(\mathbb{R})$ that it meets (see Lemma 3.6). Here we identify $\mathbb{R}^2 = (\mathbb{P}^2 \setminus L)(\mathbb{R})$.

Remark 3.2. A priori, $\mathrm{QType}_L(B)$ depends on the choice of the polynomial f cutting out Q, as well as on the choice of linear form ℓ determining ∂_L. However, changing either of these choices changes both $\partial_L f(z_i)$ by the same non-zero scalar. This shows that $\mathrm{sign}(\partial_L f(z_1) \cdot \partial_L f(z_2))$ is well-defined.

For an admissible line $L \subset \mathbb{P}^2$ let

$$s_L(Q) := \sum_{B \text{ real bitangent}} \mathrm{QType}_L(B)$$

denote the signed count (relative to L) of bitangents.

Definition 3.3. Let $L \subset \mathbb{P}^2$ a real line and B a split bitangent such that $B \cap L \cap Q = \varnothing$. The *grate* $g_L(B)$ of B with respect to L is the convex hull of $B \cap Q$ in $\mathbb{R}^2 = (\mathbb{P}^2 \backslash L)(\mathbb{R})$. We say that the grate of B is *positive* resp. *negative* if $\mathrm{QType}_L(B) = +1$ or $\mathrm{QType}_L(B) = -1$ respectively.

The statement of the next lemma can be found in [LV21, p. 15].

Lemma 3.4. *Let B be a split bitangent and L_1, L_2 two real lines with $B \cap L_i \cap Q = \varnothing$ for $i = 1, 2$. Then one has*

$$\mathrm{QType}_{L_2}(B) = \begin{cases} \mathrm{QType}_{L_1}(B) & \text{if } L_2 \cap g_{L_1}(B) = \varnothing \\ -\mathrm{QType}_{L_1}(B) & \text{if } L_2 \cap g_{L_1}(B) \neq \varnothing. \end{cases}$$

From this one can deduce the following, which is [LV21, Equ. (7)].

Lemma 3.5. *Let $L, L' \subset \mathbb{P}^2$ two admissible lines. For $\epsilon \in \{\pm 1\}$ we denote by $\sigma(L, L', \epsilon)$ the number of split bitangents B with $\mathrm{QType}_L(B) = \epsilon$ such that L' intersects $g_L(B)$. Then*

$$s_{L'}(Q) = s_L(Q) - 2 \cdot (\sigma(L, L', +1) - \sigma(L, L', -1)).$$

From now on we fix an admissible line L_0 which is disjoint from $Q(\mathbb{R})$. By [LV21, Theorem 1] we have $s_{L_0}(Q) = 4$. Thus for any admissible line L we have that $s_L(Q)$ is even by Lemma 3.5. Since by [LV21, Proposition 4.3] we also have $s_L(Q) \geq 0$, it only remains to show that $s_L(Q) \leq 8$. By Lemma 3.5 this is equivalent to proving that

$$\sigma(L_0, L, -1) - \sigma(L_0, L, +1) \leq 2. \tag{3.1}$$

The general strategy proceeds in two cases. First, if our quartic has at most 8 real bitangents, then Theorem 1.1 is trivially true. Second, in the case that Q has more than 8 real bitangents, we will, for each two connected components Q_1, Q_2 of $Q(\mathbb{R})$, study the split bitangents meeting Q_1 and Q_2 separately. We first show that there are exactly two positive and two negative grates connecting Q_1 and Q_2. We will then show that if a line meets among those four grates more negative grates than positive grates, it must meet Q_1 and Q_2. By Bézout's theorem, this line does not meet any other components of $Q(\mathbb{R})$. This yields the desired upper bound.

To begin, we characterize split bitangents determining negative grates as those separating a pair of connected components of $Q(\mathbb{R})$.

Lemma 3.6. *Let B be a split bitangent and L a real line with $L \cap B \cap Q = \varnothing$. Let V_1, V_2 the closures of the two connected components of $(\mathbb{P}^2 \setminus (L \cup B))(\mathbb{R})$. Then every connected component of $Q(\mathbb{R})$ is contained in V_1 or in V_2. We have $\mathrm{QType}_L(B) = -1$ if and only if B intersects a connected component contained in V_1 and one contained in V_2.*

Proof. It is clear that if B intersects Q in only one point, then $\mathrm{QType}_L(B) = +1$. Thus we may assume that B intersects $Q(\mathbb{R})$ in two distinct points z_1 and z_2. Let $P \in (\mathbb{P}^2 \setminus L)(\mathbb{R})$ any point that is not contained in the interior of any oval of $Q(\mathbb{R})$ (i.e. P is contained in the non-orientable connected component of $(\mathbb{P}^2 \setminus Q)(\mathbb{R})$). We identify $\mathbb{R}^2$ with $(\mathbb{P}^2 \setminus L)(\mathbb{R})$ and let F a polynomial on $\mathbb{R}^2$ defining Q. Without loss of generality we can assume that $F(P) < 0$. Then the normal vector of Q at z_i, which is the gradient of F at z_1, points towards the interior of the component Q_i containing z_i. Thus the normal vectors at z_1 and z_2 point towards different connected components of $\mathbb{R}^2 \setminus B$ if and only if V_1 and V_2 each contain one of the components Q_1 and Q_2. □

Corollary 3.7. *Let L be a split bitangent to a real smooth plane quartic Q. If L only meets one connected component of $Q(\mathbb{R})$, then the grate associated to L is positive.*

Proof. By Lemma 3.6, the QType of a split bitangent can only be negative if the split bitangent meets two distinct connected components of $Q(\mathbb{R})$. □

We may thus restrict our attention to split bitangents meeting two distinct connected components of $Q(\mathbb{R})$. Let s be the number of connected components of $Q(\mathbb{R})$ and let $a = 1$ if $Q(\mathbb{C}) \setminus Q(\mathbb{R})$ is connected and $a = 0$ otherwise. By Harnack's theorem one has $0 \leq s \leq 4$. If $s = 4$, then $a = 0$ and if $s = 0$, then $a = 1$. Finally, if s is odd, then $a = 1$ [GH81, Prop. 3.1]. The total number of real bitangents equals $4 \cdot (2^{s-1} - 1 + a)$ if $s > 0$ and 4 if $s = 0$ [GH81, Prop. 5.1]. This implies in particular that if $s < 3$ there are not more than 8 real bitangents. Thus we can further restrict our attention to the case $s \geq 3$. Then the next lemma shows that there are always four bitangents meeting any pair of two distinct connected components. Moreover, these four split bitangents determine two positive grates and two negative grates.

Lemma 3.8. *Assume that $s \geq 3$. Let Q_1, Q_2 be two distinct connected components of $Q(\mathbb{R})$. The following are true:*

(a) There are exactly four split bitangents that meet both Q_1 and Q_2.

(b) With respect to L_0 the four split bitangents meeting Q_1 and Q_2 determine two positive grates G_1^+, G_2^+ and two negative grates G_1^-, G_2^-.

(c) The two positive grates G_1^+, G_2^+ are edges of the convex hull of $Q_1 \cup Q_2$.

(d) The two negative grates G_1^-, G_2^- intersect.

Proof. Let $Q_1, \ldots, Q_s$ the connected components of $Q(\mathbb{R})$. We first claim that there is a line L_1 which does not intersect $Q(\mathbb{R})$ such that Q_1 and Q_2 are contained in two different connected components of $(\mathbb{P}^2 \setminus (L_0 \cup L_1))(\mathbb{R})$. Let ω_0 be a linear form whose zero set is L_0. Since Q is embedded canonically, we can regard ω_0 as a holomorphic differential on Q which has no real zeros. As such it induces an orientation Ω_0 of $Q(\mathbb{R})$. [Kum19, Cor. 2.2] says that every but at most one orientation of $Q(\mathbb{R})$ which agrees on Q_1 with Ω_0 is induced by a holomorphic differential on Q without real zeros. Since $s \geq 3$ there are at least two orientations on $Q(\mathbb{R})$ that agree on Q_1 with Ω_0 but differ from Ω_0 on Q_2. Thus at least one of them is induced by a holomorphic differential ω_1 without real zeros. This means that the rational function $\frac{\omega_0}{\omega_1}$ is positive on Q_1 and negative on Q_2. Thus Q_1 and Q_2 are contained in two different connected components of $(\mathbb{P}^2 \setminus (L_0 \cup L_1))(\mathbb{R})$ where L_1 is the line defined as the zeros set of ω_1 regarded as linear form on $\mathbb{P}^2$.

Now we consider $Q(\mathbb{R})$ as a subset of $\mathbb{R}^2 = (\mathbb{P}^2 \setminus L_0)(\mathbb{R})$. We have shown that the components Q_1 and Q_2 are strictly separated by the line L_1 in $\mathbb{R}^2$. Thus we can apply

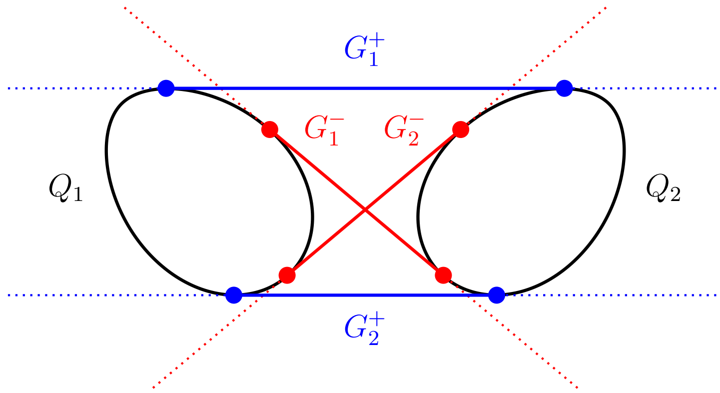

FIGURE 2. Grates meeting two components

[Kum19, Prop. 3.2] to the convex hulls of Q_1 and Q_2 which shows that there are (up to a nonzero scalar multiple) exactly two affine linear functions that are nonnegative on $Q_1 \cup Q_2$ but zero in at least one point on each Q_1 and Q_2. Thus there are exactly two split bitangents B with $\mathrm{QType}(B) = +1$ that meet both Q_1 and Q_2. Their grates are necessarily edges of the convex hull of $Q_1 \cup Q_2$.

For proving (a), (b) and (c) it remains to show that the total number of real bitangents meeting Q_1 and Q_2 is four. Recall that a *semi-orientation* is an equivalence class of orientations modulo global reversion. Thus there are exactly 2^{s-1} semiorientations on $Q(\mathbb{R})$. First consider the case $s = 3$. Then we necessarily have $a = 1$ and [Kum23, Thm. 6.9] says that the number of real bitangents which meet Q_1 and Q_2 is the same as the number of semi-orientations on $Q(\mathbb{R})$. Since $s = 3$, there are four semi-orientations which implies the claim. Now consider the case $s = 4$. Then we necessarily have $a = 0$ and [Kum23, Thm. 6.11] says that the number of real bitangents which meet Q_1 and Q_2 is the same as the number of semi-orientations on $Q(\mathbb{R})$ that restrict to a certain fixed semi-orientation on $Q_3 \cup Q_4$. Since $s = 4$, there are four such semi-orientations which implies the claim.

For part (d) let α_i be a linear form whose zero set is the bitangent B_i such that $g_{L_0}(B_i) = G_i^-$ for $i = 1, 2$. Since $\mathrm{QType}_{L_0}(B_i) = -1$, we have by Lemma 3.6 that Q_1 and Q_2 are contained in the closure of different connected components of $(\mathbb{P}^2 \setminus (L_0 \cup B_i))(\mathbb{R})$. After replacing α_i by $-\alpha_i$ if necessary, we can thus assume that $\frac{\alpha_i}{\omega_0}$ is nonnegative on Q_1 and nonpositive on Q_2. Thus $\frac{\alpha_1}{\alpha_2}$ is nonnegative on $Q_1 \cup Q_2$. This shows that Q_1 and Q_2 are contained in the closure of the same connected component of $(\mathbb{P}^2 \setminus (B_1 \cup B_2))(\mathbb{R})$ which implies $\mathrm{QType}_{B_1}(B_2) = +1$ and $\mathrm{QType}_{B_2}(B_1) = +1$. By Lemma 3.4 and because $\mathrm{QType}_{L_0}(B_i) = -1$ for $i = 1, 2$, this shows that $B_1 \cap g_{L_0}(B_2) \neq \varnothing$ and $B_2 \cap g_{L_0}(B_1) \neq \varnothing$. Since B_1 and B_2 intersect in exactly one point, we must have $g_{L_0}(B_1) \cap g_{L_0}(B_2) \neq \varnothing$. $\square$

Given two connected components Q_1, Q_2 of $Q(\mathbb{R})$, we will consistently denote by G_1^+, G_2^+ and G_1^-, G_2^- the positive and negative grates, respectively, determined by the split bitangents meeting both Q_1 and Q_2.

Each split bitangent of Q_1, Q_2 that determines a negative grate intersects each split bitangent that determines a positive grate. In fact, these intersections occur within the positive grates.

Lemma 3.9. *Let B_1, B_2 be the split bitangents containing G_1^-, G_2^-. Then $|B_i \cap G_j^+| = 1$ for $i, j \in \{1, 2\}$.*

Proof. Any two bitangents to Q are distinct, so we just need to show that $|B_i \cap G_j^+| > 0$. We will call a split bitangent *positive* (respectively *negative*) if it determines a positive (respectively negative) grate. Each positive grate touches each Q_i once, so the region R between the positive grates and $Q_1 \cup Q_2$ is bounded by the Jordan curve theorem. The negative bitangents cannot cross the components Q_i, so the negative bitangents can only enter or exit R through the positive grates.

By Lemma 3.6, a split bitangent L meeting two components Q_1, Q_2 is negative if the regions bounded by Q_1 and Q_2 lie in different components of $\mathbb{R}^2 \backslash L$. In particular, a negative bitangent passes through the interior of the convex hull K of $Q_1 \cup Q_2$. By Lemma 2.4, $|B_i \cap \partial K| = 2$. The boundary ∂K consists of the positive grates G_1^+, G_2^+, and two other boundary components that are separated by the line segments S_i connecting $G_1^+ \cap Q_i$ and $G_2^+ \cap Q_i$ (see Figure 3). Since L cannot cross Q_i, negative bitangents cannot cross S_i and hence $B_i \cap \partial K \subset G_1^+ \cup G_2^+$. Thus $|B_i \cap G_1^+| + |B_i \cap G_2^+| = 2$, so $|B_i \cap G_j^+| = 1$. □

As a corollary, we find that the negative grates are contained in the interior of the convex hull of the positive grates.

Corollary 3.10. *The negative grates G_1^-, G_2^- are contained in the convex hull of $G_1^+ \cup G_2^+$.*

Proof. Lemma 3.9 states that the split bitangents containing G_1^- and G_2^- intersect the positive grates, so G_1^-, G_2^- are contained in the convex hull of $G_1^+ \cup G_2^+$. □

Lemma 3.11. *Let Q_0 a connected component of $Q(\mathbb{R}) \subset \mathbb{R}^2 = (\mathbb{P}^2 \backslash L_0)(\mathbb{R})$. Let $a, b \in Q_0$ two different points and S the line segment spanned by a and b. If a line intersects S, then it intersects Q_0.*

Proof. Let J the closure of one of the two connected components of $Q_0 \setminus \{a, b\}$. This is a Jordan arc with the same endpoints as S. Thus by Corollary 2.3, any line meeting S also meets $J \subset Q_0$. □

We now show that if a line meets a negative grate, it meets at least two of Q_1, Q_2, G_1^+, G_2^+.

Lemma 3.12. *Let $L \subset \mathbb{P}^2$ be an admissible line. If L meets one of the negative grates of Q_1, Q_2, then L meets at least two of Q_1, Q_2, G_1^+, G_2^+.*

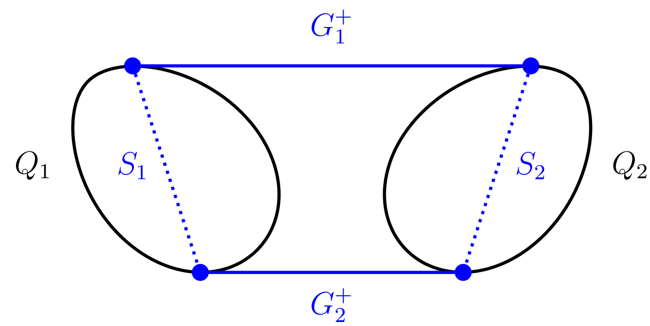

FIGURE 3. Quadrilateral from positive grates

Proof. Let R be the convex hull of $G_1^+ \cup G_2^+$. Note that R is a convex quadrilateral with edges S_1, S_2, G_1^+ and G_2^+, where S_i is the line segment connecting $G_1^+ \cap Q_i$ and $G_2^+ \cap Q_i$ (see Figure 3). If L meets a negative grate, then L meets at least two edges of R by Lemma 2.5. By Lemma 3.11 any line meeting S_i also meets Q_i. Thus if L meets at least two of S_1, S_2, G_1^+, G_2^+, then L meets at least two of Q_1, Q_2, G_1^+, G_2^+. $\square$

As a final ingredient, we show that if a line meets both negative grates, it meets either both Q_1 and Q_2 or both G_1^+ and G_2^+.

Lemma 3.13. *Let $L \subset \mathbb{P}^2$ be an admissible line. Assume that L meets both G_1^- and G_2^-. Then either L meets both Q_1 and Q_2, or L meets both G_1^+ and G_2^+.*

Proof. See Figure 4 for an illustration of some of the objects appearing in this proof. Let R be the convex hull of $G_1^- \cup G_2^-$. The negative grates intersect (necessarily within R°) by Lemma 3.8, which implies that the negative grates G_1^-, G_2^- are the diagonals of the convex quadrilateral R. Let

- E be the line segment connecting $G_1^- \cap Q_1$ and $G_2^- \cap Q_2$,
- E' be the line segment connecting $G_1^- \cap Q_2$ and $G_2^- \cap Q_1$,
- F be the line segment connecting $G_1^- \cap Q_1$ and $G_2^- \cap Q_1$, and
- F' be the line segment connecting $G_1^- \cap Q_2$ and $G_2^- \cap Q_2$.

These are the four edges of R and by Lemma 2.6, L either passes through both E, E' or through both F, F'. If L passes through both F and F', then by Lemma 3.11 L also intersects Q_1 and Q_2.

Now suppose that L passes through both E and E'. Let $x = G_1^- \cap G_2^-$ be the intersection of the diagonals of R. Let X be the set of all points lying on some line through x and E. Since L intersects E, E', Lemma 2.7 implies that L is contained in $X \cup R$. By Lemma 3.9, the positive grates G_1^+, G_2^+ bound the quadrilateral R within $X \cup R$. Any line in $X \cup R$ must pass through this bounded portion, so L must pass through G_1^+ and G_2^+. $\square$

We can now prove Theorem 1.1.

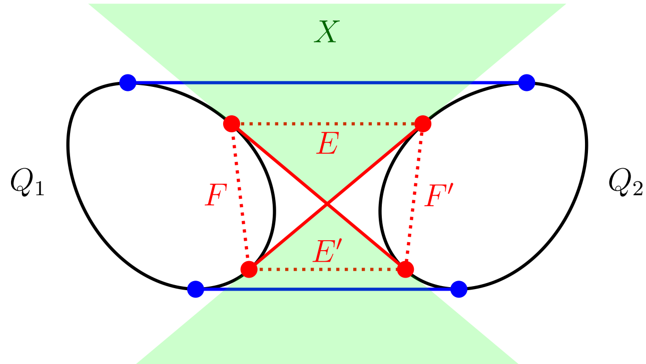

FIGURE 4. Meeting two negative grates

Proof of Theorem 1.1. Recall that we are trying to show that

$$\sigma(L_0, L, -1) - \sigma(L_0, L, +1) \leq 2,$$

where L_0 is a line disjoint from $Q(\mathbb{R})$ and L is any admissible line (see inequality 3.1). In other words, we want to show that any admissible line can meet at most 2 more negative grates than positive grates. Let $Q_1, \ldots, Q_s \subset Q(\mathbb{R})$ be the $s \geq 3$ distinct connected components. For $1 \leq i < j \leq s$ and $\epsilon \in \{\pm 1\}$ we denote by $\sigma_{ij}(L_0, L, \epsilon)$ the number of split bitangents B with $\mathrm{QType}_{L_0}(B) = \epsilon$ such that L intersects $g_{L_0}(B)$ and such that B intersects both Q_i and Q_j. Further let $\sigma_{ij}(L_0, L) = \sigma_{ij}(L_0, L, -1) - \sigma_{ij}(L_0, L, +1)$. By Corollary 3.7 we have

$$\sigma(L_0, L, -1) - \sigma(L_0, L, +1) \leq \sum_{1 \leq i < j \leq s} \sigma_{ij}(L_0, L). \tag{3.2}$$

For each $i \neq j$, let $G_{ij}^+, G_{ij}'^+$ be the positive grates and $G_{ij}^-, G_{ij}'^-$ be the negative grates associated to Q_i and Q_j. We have $\sigma_{ij}(L_0, L) > 0$ only if

(i) L meets exactly one of $G_{ij}^-, G_{ij}'^-$ and neither of $G_{ij}^+, G_{ij}'^+$, or

(ii) L meets both $G_{ij}^-, G_{ij}'^-$ and at most one of $G_{ij}^+, G_{ij}'^+$.

In case (i), L meets both Q_i and Q_j by Lemma 3.12. In case (ii), L meets both Q_i and Q_j by Lemma 3.13. Thus any case $\sigma_{ij}(L_0, L) > 0$ only if L meets both Q_i and Q_j.

However L can meet at most two of $Q_1, \ldots, Q_s$. Indeed, if $L \cap Q_i$ is nonempty, then either L is tangent to Q_i (and hence meets Q_i with multiplicity at least 2) or L passes through the region bounded by Q_i (and hence intersects Q_i at least twice). Thus L is either disjoint from Q_i or meets Q_i with multiplicity at least 2, so Bézout's theorem implies that L meets at most two connected components of $Q(\mathbb{R})$. Therefore, at most one summand on the right-hand side of Equation 3.2 is positive. Furthermore, by Lemma 3.8 we have $\sigma_{ij}(L_0, L, -1) \leq 2$ and thus $\sigma_{ij}(L_0, L) \leq 2$ for all $1 \leq i < j \leq s$. This gives the desired upper bound of 2. □

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