MOTIVIC CONFIGURATIONS ON THE LINE

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Abstract. For each configuration of rational points on the affine line, we define an operation on the group of unstable motivic homotopy classes of endomorphisms of the projective line. We also derive an algebraic formula for the image of such an operation under Cazanave and Morel's unstable degree map, which is valued in an extension of the Grothendieck–Witt group. In contrast to the topological setting, these operations depend on the choice of configuration of points via a discriminant. We prove this by first showing a local-to-global formula for the global unstable degree as a modified sum of local terms. We then use an anabelian argument to generalize from the case of local degrees of a global rational function to the case of an arbitrary collection of endomorphisms of the projective line.

1. Introduction

In topology, May's recognition principle characterizes loop spaces as algebras over the little cubes operad [May72], which is defined by operations coming from configuration spaces of Euclidean space. An analog of May's recognition principle for $\mathbb{P}^1$-loop spaces in unstable motivic homotopy theory has been sought for the last quarter century. We offer some thoughts on this question by defining a family of operations $\sum_D$ on the $\mathbb{P}^1$-loop space $\Omega_{\mathbb{P}^1}\mathbb{P}^1$. We construct these operations in terms of the configuration space of rational points in the affine line — indeed, the subscript D refers to such a configuration of points. In contrast to the topological setting, these operations depend on the set of points D via a sort of discriminant.

Let k be a field, and let $D = \{r_1, \ldots, r_n\}$ be a subset of $\mathbb{A}^1_k(k)$ with $r_i \neq r_j$ for $i \neq j$. We define the *D-pinch map* (see Definition 4.2) as the composite

$$\curlyvee_D : \mathbb{P}^1_k \xrightarrow{c_D} \frac{\mathbb{P}^1_k}{\mathbb{P}^1_k - D} \xrightarrow{\cong} \bigvee_{i=1}^{n} \frac{\mathbb{P}^1_k}{\mathbb{P}^1_k - \{r_i\}} \xleftarrow{\simeq} \bigvee_{i=1}^{n} \mathbb{P}^1_k,$$

where c_D is the collapse map induced by the inclusion $\mathbb{P}^1_k - D \hookrightarrow \mathbb{P}^1_k$, the second map is a canonical isomorphism of motivic spaces resulting from purity, and the last equivalence is a wedge of collapse maps coming from the inclusions $\mathbb{P}^1_k - \{r_i\} \hookrightarrow \mathbb{P}^1_k$. For endomorphisms $f_1, \ldots, f_n : \mathbb{P}^1_k \to \mathbb{P}^1_k$ in the unstable motivic homotopy category, we define the *D-sum* (see Definition 4.4) to be

$$\sum_D (f_1, \ldots, f_n) := \vee_i f_i \circ \curlyvee_D.$$

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Note that $\sum_D(f_1,\ldots,f_n)$ is again an endomorphism of the motivic space $\mathbb{P}^1_k$. Morel proved that such endomorphisms can be understood in terms of quadratic forms: he defined an analog of the Brouwer degree [Mor06], which is a morphism from the ring of endomorphisms of the sphere $S^n\wedge\mathbb{G}_m^{\wedge n}\simeq\mathbb{P}^n_k/\mathbb{P}^{n-1}_k$ to the Grothendieck–Witt ring $\mathrm{GW}(k)$ of isomorphism classes of non-degenerate symmetric bilinear forms over a field k. In dimensions 2 and greater, Morel's degree map is an isomorphism. In dimension 1, the degree is surjective but not injective. Morel [Mor12, Theorem 7.36] also computed

$$[\mathbb{P}^1_k,\mathbb{P}^1_k]\cong\mathrm{GW}(k)\times_{k^\times/(k^\times)^2}k^\times, \tag{1.1}$$

and Cazanave [Caz12] gave an explicit formula for this isomorphism. Let $\mathrm{GW}^u(k):=\mathrm{GW}(k)\times_{k^\times/(k^\times)^2}k^\times$, which we call the *unstable Grothendieck–Witt group*. Let

$$\deg^u:[\mathbb{P}^1_k,\mathbb{P}^1_k]\to\mathrm{GW}^u(k)$$

denote the *unstable degree*. Our main theorem is a characterization of the D-sum in terms of its image under $\deg^u$.

Theorem 1.1. *Let $D=\{r_1,\ldots,r_n\}\subset\mathbb{A}^1_k(k)$. For any unstable pointed $\mathbb{A}^1$-homotopy classes of maps $f_1,\ldots,f_n\in[\mathbb{P}^1,\mathbb{P}^1]$, we have*

$$\deg^u(\sum_D(f_1,\ldots,f_n))=\Big(\bigoplus_{i=1}^n\beta_i,\prod_{i=1}^n d_i\cdot\prod_{i<j}(r_i-r_j)^{2m_im_j}\Big),$$

where $(\beta_i,d_i)=\deg^u(f_i)$ and $m_i=\operatorname{rank}\deg^u(f_i)$ for each i.

The proof of Theorem 1.1 proceeds in two steps. The first step is to give a local-to-global formula for the unstable $\mathbb{A}^1$-degree of a rational function. To this end, we develop an unstable analog of the local $\mathbb{A}^1$-degree [KW19] and apply algebraic methods due to Cazanave [Caz12]. As a result, we find that Theorem 1.1 holds when $f_1,\ldots,f_n$ represent the unstable local degrees of a rational function whose vanishing locus is $\{r_1,\ldots,r_n\}$.

Theorem 1.2. *Let f/g be a pointed rational function with vanishing locus $\{r_1,\ldots,r_n\}\subset\mathbb{A}^1_k(k)$. For each i, let $\deg^u_{r_i}(f/g)=(\beta_i,d_i)$ and $\operatorname{rank}\beta_i=m_i$. Then*

$$\deg^u(f/g)=\Big(\bigoplus_{i=1}^n\beta_i,\prod_{i=1}^n d_i\cdot\prod_{i<j}(r_i-r_j)^{2m_im_j}\Big). \tag{1.2}$$

Theorem 1.2 will serve as the base case of an induction argument for Theorem 1.1. While carrying out this first step, we prove a few results that are of independent interest; we will mention these momentarily.

The second step to proving Theorem 1.1 is an inductive argument that uses results of Morel on the fundamental group sheaf $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$. Morel showed that $\mathbb{P}^1$ is *anabelian* in $\mathbb{A}^1$-homotopy theory [Mor12, Remark 7.32], in the sense that the $\mathbb{A}^1$-fundamental group yields a group isomorphism

$$[\mathbb{P}^1_k,\mathbb{P}^1_k]\cong\mathrm{End}(\pi_1^{\mathbb{A}^1}(\mathbb{P}^1_k)(k)).$$

Here, we borrow the term *anabelian* from Grothendieck's anabelian program in étale homotopy theory [Gro97].

As previously mentioned, the first step of our proof of Theorem 1.1 involves defining the *unstable local* $\mathbb{A}^1$*-degree.*

Definition 1.3. Let $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ be a pointed rational map. If x is a closed point such that $f(x) = 0$, then the *unstable local degree* of f at x is the image $\deg^u_x(f) \in \mathrm{GW}^u(k)$ of the map

$$\mathbb{P}^1_k \to \frac{\mathbb{P}^1_k}{\mathbb{P}^1_k - \{x\}} \to \frac{\mathbb{P}^1_k}{\mathbb{P}^1_k - \{0\}} \simeq \mathbb{P}^1_k.$$

Here the last equivalence is the one given by the crushing map $\mathbb{P}^1 \to \frac{\mathbb{P}^1_k}{\mathbb{P}^1_k - \{0\}}$. Theorem 1.2 can be thought of as a Poincaré–Hopf theorem relating the global unstable degree to its local counterparts. We also give an explicit formula for the unstable local degree at rational points in terms of a "higher residue."

Theorem 1.4. *Let f/g be a pointed rational function. Let $r \in \mathbb{A}^1_k(k)$ be a root of f of multiplicity m. Then there exists $a \in k^\times$ such that*

$$\frac{g(x)}{f(x)} = \frac{a}{(x-r)^m} + \sum_{i>-m} a_i (x-r)^i,$$

and we have

$$\deg^u_r(f) = \underbrace{\begin{pmatrix} * & * & \cdots & * & a \\ * & * & \cdots & a & 0 \\ \vdots & \vdots & ⋰ & \vdots & \vdots \\ * & a & \cdots & 0 & 0 \\ a & 0 & \cdots & 0 & 0 \end{pmatrix}}_{m \times m}.$$

1.1. **Outline.** We review some relevant terminology and notation in Section 2. In Section 3, we define the unstable local $\mathbb{A}^1$-degree and derive an algebraic formula for it under nice hypotheses. In Section 4, we define the D-sum $\sum_D$ and prove that the unstable $\mathbb{A}^1$-degree satisfies a local-to-global principle with respect to $\sum_D$.

We take a slight detour in Section 5, where we define a generalization of the polynomial discriminant (which we call the *duplicant*). Code supporting our analysis of duplicants can be found in Appendix A. Our aside on duplicants is utilized in Section 6, where we prove Theorem 1.2 (as Proposition 6.5). Most of the techniques for this proof boil down to (somewhat involved) linear algebra.

In Section 7, we prove Theorem 1.1 by proving the requisite details on $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$ and setting up an induction argument with Theorem 1.2 as the base case.

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2. Terminology and notation

We will frequently work with *pointed rational maps*, which are rational functions $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ such that $f(\infty) = \infty$. We will denote the unstable motivic homotopy category of pointed spaces over a field k by $\mathcal{H}_\bullet(k)$. Given two pointed motivic spaces X and Y, we denote the set of pointed $\mathbb{A}^1$-homotopy classes of maps $X \to Y$ by $[X, Y]$. We will really only need to consider the case of $X = Y = \mathbb{P}^1_k$.

2.1. **Unstable Grothendieck–Witt groups.** Define the unstable Grothendieck–Witt group

$$\mathrm{GW}^u(k) := \mathrm{GW}(k) \times_{k^\times/(k^\times)^2} k^\times.$$

We refer to the $\mathrm{GW}(k)$ and $k^\times$ factors of $\mathrm{GW}^u(k)$ as the *stable* and *unstable parts*, respectively. The group structure on $\mathrm{GW}^u(k)$ is given by $(\beta_1, b_1)+(\beta_2, b_2) = (\beta_1+\beta_2, b_1b_2)$ (or in words, by taking direct sums of the stable parts and multiplying the unstable parts). We wish to describe $\mathrm{GW}^u(k)$ in terms of generators and relations. To this end, we recall the usual presentation of $\mathrm{GW}(k)$.

Proposition 2.1. *Let k be a field. Given $a \in k^\times$, let $\langle a\rangle$ be the isomorphism class of the bilinear form $(x, y) \mapsto axy$. As a group, $\mathrm{GW}(k)$ is isomorphic to the group generated by $\{\langle a\rangle : a \in k^\times\}$ modulo the following relations:*

(i) $\langle ab^2\rangle = \langle a\rangle$ *for all* $a, b \in k^\times$.

(ii) $\langle a\rangle + \langle b\rangle = \langle a + b\rangle + \langle ab(a + b)\rangle$ *for all* $a, b \in k^\times$ *such that* $a + b \neq 0$.

(iii) $\langle a\rangle + \langle -a\rangle = \langle 1\rangle + \langle -1\rangle$ *for all* $a \in k^\times$.

Moreover, one recovers $\mathrm{GW}(k)$ as a ring by imposing the further relation:

(iv) $\langle a\rangle\langle b\rangle = \langle ab\rangle$ *for all* $a, b \in k^\times$.

Remark 2.2. Relations (i) and (ii) actually imply relation (iii). Because of its ubiquity, we define the *hyperbolic form* $\mathbb{H} := \langle 1\rangle + \langle -1\rangle$.

Following the stable case, we can give a presentation of the unstable Grothendieck–Witt group in terms of generators and relations.

Proposition 2.3. *Let k be a field. Given $a \in k^\times$, let $\langle a\rangle^u := (\langle a\rangle, a) \in \mathrm{GW}^u(k)$. As a group, $\mathrm{GW}^u(k)$ is isomorphic to the group generated by $\{\langle a\rangle^u : a \in k^\times\}$ modulo the following relations:*

(i') $\langle ab^2\rangle^u = \langle a\rangle^u + \langle b\rangle^u - \langle 1/b\rangle^u$ *for all* $a, b \in k^\times$.

(ii') $\langle a\rangle^u + \langle b\rangle^u = \langle 1/(a + b)\rangle^u + \langle ab(a + b)\rangle^u$ *for all* $a, b \in k^\times$ *such that* $a + b \neq 0$.

Proof. By definition, each element of $\mathrm{GW}^u(k)$ is of the form (β, d), where $\beta \in \mathrm{GW}(k)$ and $d \in k^\times$ such that $d \equiv \operatorname{disc}\beta \mod (k^\times)^2$. Writing $\beta = \sum_{i=1}^n \langle a_i\rangle - \sum_{j=1}^m \langle b_j\rangle$ in $\mathrm{GW}(k)$, we have $d = c^2(\prod_i a_i)(\prod_j b_j^{-1})$ for some $c \in k^\times$. Since

$$\begin{aligned}\langle c\rangle^u - \langle 1/c\rangle^u &= (\langle c\rangle, c) - (\langle 1/c\rangle, 1/c)\\ &= (\langle c\rangle, c) - (\langle c\rangle, 1/c)\\ &= (0, c^2)\end{aligned}$$

by Proposition 2.1 (i), we have $(\beta, d) = \langle c\rangle^u - \langle 1/c\rangle^u + \sum_{i=1}^n \langle a_i\rangle^u - \sum_{j=1}^m \langle b_j\rangle^u$. That is, $\mathrm{GW}^u(k)$ is generated by elements of the form $\langle a\rangle^u$.

There are no relations on $\mathrm{GW}^u(k)$ imposed by the unstable factor $k^\times$, so we only need the additive relations on the stable factor given in Proposition 2.1. Relation (i') is precisely Proposition 2.1 (i) when restricted to elements of the form $\langle a\rangle^u$. For relation (ii'), we have $\langle 1/(a+b)\rangle = \langle a+b\rangle$ in $\mathrm{GW}(k)$. It remains to check that the unstable factors agree, which is merely the computation $ab = \frac{1}{a+b}\cdot ab(a+b)$. □

Remark 2.4. If we present $\mathrm{GW}(k)$ as a ring, it turns out that Proposition 2.1 (i), (ii), and (iv) imply relation (iii). Since we do not consider any ring structure on $\mathrm{GW}^u(k)$, we do not have an analog of Proposition 2.1 (iv) for Proposition 2.3. Consequently, there is no relation analogous to Proposition 2.1 (iii) that needs to be imposed on $\mathrm{GW}^u(k)$. However, one can calculate that $\langle 1/a\rangle^u + \langle -a\rangle^u = \langle 1\rangle^u + \langle -1\rangle^u$ for all $a \in k^\times$. We denote this *unstable hyperbolic form* by $\mathbb{H}^u$.

2.2. **Bézoutians.** We will briefly recall some details about univariate Bézoutians, which provide an algebraic formula for the unstable degree by [Caz12].

Definition 2.5. Given a pointed rational function $f/g : \mathbb{P}^1_k \to \mathbb{P}^1_k$, the *Bézoutian polynomial* of f/g is defined to be

$$\mathrm{Béz}(f/g) := \frac{f(X)g(Y) - f(Y)g(X)}{X - Y} \in k[X, Y].$$

The *Bézoutian matrix with respect to the monomial basis* is the matrix

$$\mathrm{Béz}^{\mathrm{mon}}(f/g) := (a_{ij})_{i,j=0}^m,$$

where $a_{ij} \in k$ are such that $\mathrm{Béz}(f/g) = \sum_{i,j} a_{ij}X^iY^j$.

Remark 2.6. The term *monomial basis* in Definition 2.5 refers to the monomial basis $\{x^i\}_{i,j}$ of the k-algebra $Q(f/g) := k[x, \frac{1}{g}]/(\frac{f}{g})$. The Bézoutian can be viewed as an element of $Q(f/g) \otimes_k Q(f/g)$ under the isomorphism

$$\begin{aligned}Q(f/g) \otimes_k Q(f/g) &\to k[X, Y, {}^1\!/_{g(X)}, {}^1\!/_{g(Y)}]/({}^{f(X)}\!/_{g(X)}, {}^{f(Y)}\!/_{g(Y)})\\ a(x) \otimes b(x) &\mapsto a(X)b(Y).\end{aligned}$$

The Bézoutian matrix with respect to the monomial basis is then the coefficient matrix of the Bézoutian polynomial in the basis $\{X^iY^j\}_{i,j}$.

We will also need another choice of basis for $Q(f/g)$.

Definition 2.7. Let $f/g : \mathbb{P}^1_k \to \mathbb{P}^1_k$ be a pointed rational function with rational root r of order m. Consider the k-algebra

$$Q_r(f/g) := \frac{k[x, {}^1\!/\!_g]_{(x-r)}}{(f, {}^1\!/\!_g)},$$

The *local Newton basis* of $Q_r(f/g)$ is the basis

$$B_r^{\mathrm{Nwt}}(f/g) := \left\{ \frac{f}{g \cdot (x-r)}, \frac{f}{g \cdot (x-r)^2}, \ldots, \frac{f}{g \cdot (x-r)^m} \right\}.$$

If all roots of f are k-rational, then we define the *(global) Newton basis* of $Q(f/g)$ as

$$B^{\mathrm{Nwt}}(f/g) := \bigcup_{r \in f^{-1}(0)} B_r^{\mathrm{Nwt}}(f/g).$$

Remark 2.8. Any symmetric non-degenerate matrix M over a field k represents a symmetric non-degenerate bilinear form over k. Given such a matrix M, we will also denote the isomorphism class of the bilinear form that it represents by $M \in \mathrm{GW}(k)$.

Cazanave computes the unstable global degree in terms of the Bézoutian with respect to the monomial basis [Caz12, Theorem 3.6].

Theorem 2.9 (Cazanave)**.** *There is a group isomorphism*

$$\deg^u : ([\mathbb{P}^1_k, \mathbb{P}^1_k], \oplus^{\mathrm{N}})^{\mathrm{gp}} \to \mathrm{GW}^u(k)$$

given by $\deg^u(f/g) = (\mathrm{Béz}^{\mathrm{mon}}(f/g), \det \mathrm{Béz}^{\mathrm{mon}}(f/g))$.

Here, the superscript gp denotes group completion (which is necessary, as the Bézoutian bilinear form only realizes elements of non-negative rank). The symbol $\oplus^{\mathrm{N}}$ is Cazanave's *naïve sum*, which is a monoid structure on the set $[\mathbb{P}^1_k, \mathbb{P}^1_k]$. We will recall the definition of $\oplus^{\mathrm{N}}$ in Definition 6.1 when it becomes more relevant for us.

Remark 2.10. Note that $\mathrm{Béz}(cf/cg) = c^2\mathrm{Béz}(f/g)$. This c^2 factor does not cause any inconsistencies in the stable setting, as $\langle c^2 \rangle = \langle 1 \rangle$ in $\mathrm{GW}(k)$. However, this c^2 factor would cause $(\mathrm{Béz}^{\mathrm{mon}}(f/g), \det \mathrm{Béz}^{\mathrm{mon}}(f/g))$ to be ill-defined in $\mathrm{GW}^u(k)$. To get a well-defined Bézoutian, we therefore always normalize f/g so that f is monic. This is the same convention used in [Caz12].

When f is a polynomial morphism, $\deg^u(f)$ is fully determined by the leading coefficient. Our convention that f is monic forces $\deg^u(f)$ to scale inversely rather than directly:

Proposition 2.11. *Let* $f(x) = \sum_{i=0}^n a_i x^i \in k[x]$. *Then* $\deg^u(f) \in \mathrm{GW}^u(k)$ *is presented by any matrix of the form*

$$\begin{pmatrix} * & * & \cdots & * & a_n^{-1} \\ * & * & \cdots & a_n^{-1} & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ * & a_n^{-1} & \cdots & 0 & 0 \\ a_n^{-1} & 0 & \cdots & 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 & \cdots & 0 & a_n^{-1} \\ 0 & 0 & \cdots & a_n^{-1} & * \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & a_n^{-1} & \cdots & * & * \\ a_n^{-1} & * & \cdots & * & * \end{pmatrix}. \tag{2.1}$$

Proof. Because we normalize so that f is monic, we write $f = \frac{x^n+\sum_i a_i a_n^{-1} x^i}{a_n^{-1}}$. One can readily compute that $\text{Béz}(\frac{x^n+\sum_i a_i a_n^{-1} x^i}{a_n^{-1}}) = a_n^{-1} \sum_{i+j=n-1} X^i Y^j + \sum_{\ell=1}^{n-1} a_\ell a_n^{-1} \sum_{i+j=\ell-1} X^i Y^j$, so the Bézoutian matrix with respect to the monomial basis is given by

$$\text{Béz}^{\text{mon}}(f) = \begin{pmatrix} a_1 a_n^{-1} & a_2 a_n^{-1} & \cdots & a_{n-1} a_n^{-1} & a_n^{-1} \\ a_2 a_n^{-1} & a_3 a_n^{-1} & \cdots & a_n^{-1} & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ a_{n-1} a_n^{-1} & a_n^{-1} & \cdots & 0 & 0 \\ a_n^{-1} & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

The element of $\text{GW}(k)$ determined by the matrix $\text{Béz}^{\text{mon}}(f)$ depends only on a_n^{-1} (by e.g. [KW20, Lemma 6]). Moreover, the determinant of any (anti)-triangular matrix is determined by its diagonal, so any matrix of the form Equation 2.1 determines the same class in $\text{GW}^u(k)$ as $(\text{Béz}^{\text{mon}}(f), \det \text{Béz}^{\text{mon}}(f))$. □

3. Unstable local degree

Following the stable setting, we will define the *unstable local degree* of a map of curves at a closed point with rational image.

Setup 3.1. Let X and Y be curves over k. Let $f : X \to Y$ be a morphism. Assume that $x \in X$ is a closed point such that $f(x) \in Y(k)$. Let $U \subseteq X$ and $V \subseteq Y$ be Zariski open neighborhoods of x and $f(x)$, respectively. Assume that x is isolated in its fiber, so that (shrinking U and V as necessary) f defines a map

$$\bar{f}_x : U/(U - \{x\}) \to V/(V - \{f(x)\}).$$

By excision, we can rewrite this as

$$\bar{f}_x : \mathbb{P}^1_k/(\mathbb{P}^1_k - \{x\}) \to \mathbb{P}^1_k/(\mathbb{P}^1_k - \{f(x)\}) \simeq \mathbb{P}^1_k.$$

In order to obtain an element of $[\mathbb{P}^1_k, \mathbb{P}^1_k]$, we precompose with the collapse map $c_x : \mathbb{P}^1_k \to \mathbb{P}^1_k/(\mathbb{P}^1_k - \{x\})$.

Remark 3.2. Suppose that f has vanishing locus $D = \{x_1, \ldots, x_n\}$. We can then form the collapse map

$$c_D : \mathbb{P}^1_k \to \frac{\mathbb{P}^1_k}{\mathbb{P}^1_k - D}$$

from the inclusion $\mathbb{P}^1_k - D \hookrightarrow \mathbb{P}^1_k$. There is a canonical isomorphism $\mathbb{P}^1_k/(\mathbb{P}^1_k - D) \cong \bigvee_{i=1}^n \mathbb{P}^1_k/(\mathbb{P}^1_k - \{x_i\})$ [Caz12, Lemma A.3]. The induced maps $\bar{f}_{x_i} : \mathbb{P}^1_k/(\mathbb{P}^1_k - \{x_i\}) \to \mathbb{P}^1_k$ are constructed such that the diagram

$$\begin{array}{ccc} \mathbb{P}^1_k & \xrightarrow{f} & \mathbb{P}^1_k \\ {\scriptstyle c_D}\downarrow & & \uparrow{\scriptstyle \cong} \\ \bigvee_i \frac{\mathbb{P}^1_k}{\mathbb{P}^1_k - \{x_i\}} & \xrightarrow{\vee_i \bar{f}_{x_i}} & \frac{\mathbb{P}^1_k}{\mathbb{P}^1_k - D} \end{array}$$

commutes.

Definition 3.3. Assume the notation of Setup 3.1. The *unstable local degree* of f at x is the image $\deg^u_x(f) \in \mathrm{GW}^u(k)$ of the composite $\bar{f}_x \circ c_x$ under Cazanave's isomorphism (Equation 1.1). We will sometimes find it convenient to call $\deg^u_x(f) \in \mathrm{GW}^u(k)$ the *algebraic* unstable local degree, in contrast to the *homotopical* unstable local degree $\bar{f}_x \circ c_x \in [\mathbb{P}^1_k, \mathbb{P}^1_k]$.

Note that if x is the only zero of f, then the unstable degree coincides with the unstable local degree.

Proposition 3.4. *Let $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ be a pointed rational map with $f^{-1}(0) = \{x\}$. Assume that $x \in \mathbb{A}^1_k(k)$. Then $\deg^u_x(f) = \deg^u(f)$.*

Proof. By definition of the unstable local degree, it suffices to show that the diagram

$$\begin{array}{ccc} \mathbb{P}^1_k & \xrightarrow{c_x} & \mathbb{P}^1_k/(\mathbb{P}^1_k - \{x\}) \\ \downarrow f & & \downarrow \bar{f}_x \\ \mathbb{P}^1_k & \xleftarrow[\simeq]{} & \mathbb{P}^1/(\mathbb{P}^1 - \{0\}) \end{array} \tag{3.1}$$

commutes in $\mathcal{H}_\bullet(k)$. The commutativity of Diagram 3.1 is explained in Remark 3.2 (setting $n = 1$). □

Remark 3.5. Precomposition with the collapse map should be thought of as a transfer $c_x^* : \mathrm{GW}^u(k(x)) \to \mathrm{GW}^u(k)$, where $k(x)$ is the residue field of x. When x is k-rational, the collapse map is in fact a homotopy equivalence $\mathbb{P}^1_k \simeq \mathbb{P}^1_k/(\mathbb{P}^1_k - \{x\})$ of pointed motivic spaces. Throughout this article, we will assume that x is k-rational. We will give an analysis of the *unstable transfer* c_x^* and the unstable local degree at non-rational points in future work.

3.1. **Algebraic formula for the unstable local degree.** We now give two formulas for the unstable local degree at rational points. The first formula assumes that we are computing the unstable local degree at a simple zero, in which case the local degree is given by the inverse of the derivative. This is the unstable analog of [KW19, Lemma 9].

Remark 3.6. We are working with pointed rational functions f/g, which means that $\infty \in \mathbb{P}^1_k$ is not a root of f. In other words, all roots of f lie in $\mathbb{A}^1_k = \mathbb{P}^1_k - \{\infty\}$.

Proposition 3.7. *Let $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ be a pointed rational map. Assume that $x \in \mathbb{A}^1_k(k)$ is a simple k-rational zero of f. Then $\deg^u_x(f) = \langle f'|_x^{-1}\rangle^u$.*

Proof. This is the unstable, k-rational version of [KW19, Proposition 15]. Because the proof in *loc. cit.* makes use of the stable motivic homotopy category, we need to modify the proof to hold in $\mathcal{H}_\bullet(k)$.

Because x is a simple zero of f (equivalently, f is étale at x), the induced map of tangent spaces $df_x : T_x\mathbb{P}^1_k \to f^*T_{f(x)}\mathbb{P}^1_k$ is a monomorphism. Thus df_x induces a map

$\mathrm{Th}(df_x) : \mathrm{Th}(T_x\mathbb{P}^1_k) \to \mathrm{Th}(f^*T_{f(x)}\mathbb{P}^1_k)$ of Thom spaces. Because x and $f(x)$ are k-rational, we have isomorphisms $\mathrm{Th}(T_x\mathbb{P}^1_k) \cong \mathrm{Th}(\mathcal{O}_{\mathrm{Spec}\,k}) \cong \mathrm{Th}(f^*T_{f(x)}\mathbb{P}^1_k)$ in $\mathcal{H}_\bullet(k)$, which fit into the commutative diagram

$$\begin{array}{ccc} \mathrm{Th}(T_x\mathbb{P}^1_k) & \xrightarrow{\mathrm{Th}(df_x)} & \mathrm{Th}(f^*T_{f(x)}\mathbb{P}^1_k) \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{Th}(\mathcal{O}_{\mathrm{Spec}\,k}) & \xrightarrow{f'|_x} & \mathrm{Th}(\mathcal{O}_{\mathrm{Spec}\,k}). \end{array} \tag{3.2}$$

Here, $f'|_x$ refers to the linear map $z \mapsto f'|_x \cdot z$. Note that $f'|_x \in k^\times$ since f is étale at x. The naturality of the purity isomorphism [Voe03, Lemma 2.1] yields a commutative diagram

$$\begin{array}{ccccccc} & & \mathrm{Th}(T_xU) & \xrightarrow{\mathrm{Th}(df_x)} & \mathrm{Th}(f^*T_{f(x)}V) & & \\ & & \cong\downarrow & & \downarrow\cong & & \\ \mathbb{P}^1_k & \xrightarrow{\simeq} & \frac{U}{U-\{x\}} & \xrightarrow{f|_U} & \frac{V}{V-\{f(x)\}} & \xrightarrow{\simeq} & \mathbb{P}^1_k. \end{array} \tag{3.3}$$

By stacking Diagrams 3.2 and 3.3, we find that $\deg^u_x(f) = \deg^u(z \mapsto f'|_x \cdot z)$. In other words, we have reduced computing $\deg^u_x(f)$ to computing the unstable *global* degree of a pointed rational function. We may therefore apply [Caz12] and compute $\deg^u(z \mapsto f'|_x \cdot z) = \langle f'|_x^{-1}\rangle^u$ (see Proposition 2.11). □

Now we give a more general, algebraic formula for the unstable local degree at rational points. This formula, which is the unstable analog of [KW19, Main Theorem] and [BMP23, Theorem 1.2], involves the *local Newton matrix* [KW20, Definition 7].

Definition 3.8. Let f/g be a pointed rational function. Let $r \in \mathbb{A}^1_k(k)$ be a root of f of multiplicity m. Write a partial fraction decomposition

$$\frac{g(x)}{f(x)} = \frac{A_{r,m}}{(x-r)^m} + \frac{A_{r,m-1}}{(x-r)^{m-1}} + \cdots + \frac{A_{r,1}}{x-r} + \text{higher order terms}.$$

Define the *local Newton matrix*

$$\mathrm{Nwt}_r(f/g) := \begin{pmatrix} A_{r,1} & A_{r,2} & \cdots & A_{r,m-1} & A_{r,m} \\ A_{r,2} & A_{r,3} & \cdots & A_{r,m} & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ A_{r,m-1} & A_{r,m} & \cdots & 0 & 0 \\ A_{r,m} & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

The local Newton matrix represents a class in $\mathrm{GW}^u(k)$, which we also denote by $\mathrm{Nwt}_r(f/g)$.

To prove that $\mathrm{Nwt}_r(f/g)$ computes $\deg^u_r(f/g)$, we first show that the unstable local degree is an $\mathbb{A}^1$-homotopy invariant (c.f. [KW20, Lemma 4]).

Lemma 3.9. *Let $r \in \mathbb{A}^1_k$ be a closed point. Let $\frac{f_0}{g_0}, \frac{f_1}{g_1} : \mathbb{P}^1_k \to \mathbb{P}^1_k$ be pointed rational functions such that $f_0(r) = f_1(r) = 0$. Suppose there exists an open subscheme $U \subseteq$*

$\mathbb{A}^1_k \times \mathbb{A}^1_k$ *containing* $\{r\} \times \mathbb{A}^1_k$ *and a morphism* $H : U \to \mathbb{P}^1_k$ *such that* $H(x,0) = \frac{f_0}{g_0}(x)$ *and* $H(x,1) = \frac{f_1}{g_1}(x)$. *If* $\{r\} \times \mathbb{A}^1_k$ *is a connected component of* $H^{-1}(\{0\} \times \mathbb{A}^1_k)$, *then*

$$\deg^u_r(f_0/g_0) = \deg^u_r(f_1/g_1).$$

Proof. Let Z be the union of the connected components of $H^{-1}(\{0\} \times \mathbb{A}^1_k)$ that are distinct from $\{r\} \times \mathbb{A}^1_k$. We can then write

$$\begin{aligned} \frac{U}{U - H^{-1}(0)} &= \frac{U}{U - ((\{r\} \times \mathbb{A}^1_k) \amalg Z)} \\ \text{[Caz12, Lemma A.3]} \qquad &\simeq \frac{U}{U - (\{r\} \times \mathbb{A}^1_k)} \vee \frac{U}{U - Z} \\ \text{(excision)} \qquad &\simeq \frac{\mathbb{P}^1_k \times \mathbb{A}^1_k}{\mathbb{P}^1_k \times \mathbb{A}^1_k - (\{r\} \times \mathbb{A}^1_k)} \vee \frac{U}{U - Z}. \end{aligned}$$

This implies that the morphism $\frac{U}{U - H^{-1}(0)} \to \frac{\mathbb{P}^1_k}{\mathbb{P}^1_k - \{0\}}$ induced by H is equivalent to a morphism

$$\frac{\mathbb{P}^1_k \times \mathbb{A}^1_k}{\mathbb{P}^1_k \times \mathbb{A}^1_k - (\{r\} \times \mathbb{A}^1_k)} \vee \frac{U}{U - Z} \to \frac{\mathbb{P}^1_k}{\mathbb{P}^1_k - \{0\}}. \tag{3.4}$$

Pre-composing Equation 3.4 with the natural morphisms

$$\frac{\mathbb{P}^1_k}{\mathbb{P}^1_k - \{r\}} \times \mathbb{A}^1_k \to \frac{\mathbb{P}^1_k \times \mathbb{A}^1_k}{\mathbb{P}^1_k \times \mathbb{A}^1_k - (\{r\} \times \mathbb{A}^1_k)} \to \frac{\mathbb{P}^1_k \times \mathbb{A}^1_k}{\mathbb{P}^1_k \times \mathbb{A}^1_k - (\{r\} \times \mathbb{A}^1_k)} \vee \frac{U}{U - Z}$$

gives us a naïve $\mathbb{A}^1$-homotopy from the map $\overline{(\frac{f_0}{g_0})_r}$ to $\overline{(\frac{f_1}{g_1})_r}$ (in the notation of Setup 3.1). It follows that we have a naïve homotopy from $\overline{(\frac{f_0}{g_0})_r} \circ c_r$ to $\overline{(\frac{f_1}{g_1})_r} \circ c_r$, and hence these maps determine the same element of $\mathrm{GW}^u(k)$. □

Using Lemma 3.9, we can now compute $\deg^u_r(f/g) = \mathrm{Nwt}_r(f/g)$ when r is a rational point (c.f. [KW20, Corollary 8]).

Lemma 3.10. *Let* f/g *be a pointed rational function. Let* $r \in \mathbb{A}^1_k(k)$ *be a root of* f. *Then*

$$\deg^u_r(f/g) = \mathrm{Nwt}_r(f/g).$$

Proof. Since r is a root of f of order m, there exist $A \in k^\times$ and a polynomial $f_0(x) \in k[x]$ such that $f(x) = (x-r)^m(A + (x-r)f_0(x))$. Similarly, since f/g is a pointed rational function, r is not a root of g and hence there exist $B \in k^\times$ and a polynomial $g_0(x) \in k[x]$ such that $g(x) = B + (x-r)g_0(x)$.

Now let $U = \{(x,t) \in \mathbb{P}^1_k \times \mathbb{A}^1_k : x \neq \infty \text{ and } g(x) \neq 0\}$. Then

$$H_1(x,t) = \frac{(x-r)^m(A + t(x-r)f_0(x))}{g(x)}$$

determines a morphism $H_1 : U \to \mathbb{P}^1_k$ such that $H_1(x,0) = \frac{A(x-r)^m}{g(x)}$ and $H_1(x,1) = \frac{f}{g}(x)$. This morphism satisfies the criteria of Lemma 3.9, which implies

$$\deg_r^u(f/g) = \deg_r^u(A(x-r)^m/g(x)).$$

Next, we get a morphism $H_2 : \mathbb{P}^1_k \times \mathbb{A}^1_k \to \mathbb{P}^1_k$ given by

$$H_2(x,t) = \frac{A(x-r)^m}{B + t(x-r)g_0(x)}$$

that also satisfies the criteria of Lemma 3.9. Thus

$$\deg_r^u(A(x-r)^m/g(x)) = \deg_r^u(A(x-r)^m/B).$$

Since r is the only root of $A(x-r)^m/B$, it follows from Proposition 3.4 that $\deg_r^u(f/g) = \deg^u(A(x-r)^m/B)$. We now normalize $A(x-r)^m/B = \frac{(x-r)^m}{B/A}$ and apply Proposition 2.11 to compute

$$\deg^u(\tfrac{(x-r)^m}{B/A}) = \begin{pmatrix} * & * & \cdots & * & \frac{B}{A} \\ * & * & \cdots & \frac{B}{A} & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ * & \frac{B}{A} & \cdots & 0 & 0 \\ \frac{B}{A} & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

It thus suffices to prove that $\frac{B}{A} = A_{r,m}$. Given a rational function F, let $\mathrm{Res}^m(F,r)$ denote the coefficient of $(x-r)^{-m}$ in the Laurent expansion of F about r,[1] so that $A_{r,m} = \mathrm{Res}^m(g/f, r)$. Since $f(x) = A(x-r)^m(1 + (x-r)f_0(x))$, we have

$$\frac{1}{f} = \frac{1}{A(x-r)^m} \sum_{i\geq 0} a_i(x-r)^i$$

with $a_0 \in k^\times$ and $a_i \in k$ for $i > 0$. Thus

$$\begin{aligned} A_{r,m} &= \mathrm{Res}^m\big(\frac{g}{f}, r\big) \\ &= \mathrm{Res}^m\big(\frac{B + (x-r)g_0}{A(x-r)^m} \sum_{i\geq 0} a_i(x-r)^i, r\big) \\ &= \frac{B}{A}, \end{aligned}$$

as desired. □

Remark 3.11. Lemma 3.10 corroborates Proposition 3.7. If f/g has a simple root at r, then Lemma 3.10 (in particular, its proof) implies that $\deg_r^u(f/g) = \langle \mathrm{Res}(g/f, r)\rangle^u$.

[1] One might call Res^m a *higher residue*, since Res^1 is the usual residue from complex analysis.

The standard trick for computing the residue of a simple pole tells us

$$\begin{aligned}\operatorname{Res}(g/f,r) &= \frac{g(r)}{f'(r)} \\ &= \frac{g(r)^2}{f'(r)\cdot g(r) - f(r)\cdot g'(r)} \\ &= (f/g)'(r)^{-1},\end{aligned}$$

since $f(r)=0$. Thus $\langle \operatorname{Res}(g/f,r)\rangle^u = \langle (f/g)'|_r^{-1}\rangle^u$.

4. Local-to-global principle, homotopically

Given a map $f:\mathbb{P}^1_k \to \mathbb{P}^1_k$, we are interested in understanding the relationship between the unstable degree $\deg^u(f)$ and the unstable local degrees $\deg^u_x(f)$ for $x\in f^{-1}(0)$. In particular, we would like to prove a *local-to-global principle* or *local decomposition* for $\deg^u(f)$, namely that

$$\deg^u(f) = \sum_{x\in f^{-1}(0)} \deg^u_x(f). \tag{4.1}$$

In topology, such local decompositions give rise to the Poincaré–Hopf theorem for vector bundles. A crucial aspect of Equation 4.1 is that the sum is indexed over the vanishing locus $f^{-1}(0)$ — we do not only want to express $\deg^u(f)$ in terms of simpler summands, but rather that these summands have an explicit and tractable geometric relationship to the morphism f.

In this section, we will prove a homotopical local decomposition

$$f = \sum_{x\in f^{-1}(0)} \bar{f}_x \circ c_x. \tag{4.2}$$

In Section 6, we will obtain an algebraic local decomposition $\deg^u(f) = \sum_{x\in f^{-1}(0)} \deg^u_x(f)$ by analyzing the image of Equation 4.2 in $\mathrm{GW}^u(k)$. We will also discuss Cazanave's decomposition of $\deg^u(f)$ and how it fails to be local.

Homotopically, sums of maps are given by pinching and folding. That is, given $f,g: X\to Y$, the sum $f+g$ is defined as the composite

$$X \xrightarrow{\curlyvee} X\vee X \xrightarrow{f\vee g} Y\vee Y \xrightarrow{\nabla} Y.$$

The fold is actually unnecessary for our purposes: the wedge is the coproduct in pointed spaces, so maps out of the wedge are in bijection with a set of maps out of each to a fixed target. Post-composition with the fold map would be necessary if we were working with an external wedge sum, which we will not need in this article.

Whenever X is a suspension $X\simeq S^1\wedge X'$, we can construct a pinch map as follows. Any choice of inclusion $S^0\subset S^1$ separates S^1 into two disjoint intervals; collapsing S^0 closes each of these intervals off into an S^1, with the two copies of S^1 joined together at the image of S^0 (see Figure 1). One then defines the pinch $\curlyvee : X\to X\vee X$ as

$$S^1\wedge X' \xrightarrow{\curlyvee} (S^1\vee S^1)\wedge X' \simeq (S^1\wedge X')\vee(S^1\wedge X').$$

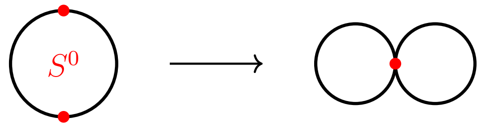


FIGURE 1. Pinching S^1

Here, the last homotopy equivalence holds in any category where smash products distribute over wedge sums, i.e. any category in which products commute with pushouts.

In order to add pointed endomorphisms of $\mathbb{P}^1$, we need a workable pinch map $\mathbb{P}^1 \to \mathbb{P}^1 \vee \mathbb{P}^1$. While $\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m$ as a motivic space, the simplicial pinch map $\mathbb{P}^1 \to \mathbb{P}^1 \vee \mathbb{P}^1$ is unwieldy from the perspective of algebraic geometry. That is, there is not an evident way to describe the simplicial pinch in terms of subschemes of $\mathbb{P}^1$. This stems from the fact that we need $\mathbb{A}^1$-invariance to realize $\mathbb{P}^1$ as a suspension:

$$\begin{array}{ccc} \mathbb{G}_m & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & S^1 \wedge \mathbb{G}_m \end{array} \qquad \simeq \qquad \begin{array}{ccc} \mathbb{G}_m & \longrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1. \end{array}$$

While the simplicial pinch map gives the usual group structure on $[\mathbb{P}^1_k, \mathbb{P}^1_k] \cong \mathrm{GW}^u(k)$ [Caz12, Lemma 3.20 and Theorem 3.21], Cazanave noticed that the collapse map can be viewed as an algebraic pinch map [Caz12, Lemma A.3]. Cazanave used these algebraic pinch maps to define the naïve sum $\oplus^{\mathrm{N}} : [\mathbb{P}^1, \mathbb{P}^1]^2 \to [\mathbb{P}^1, \mathbb{P}^1]$ [Caz12, §3.1], which give a method for decomposing global maps into "local" terms. However, as we will describe in Section 6.1, the naïvely local terms of a map $f : \mathbb{P}^1 \to \mathbb{P}^1$ fail to be truly local.

While Cazanave only considers the pinch map arising from the collapse map $c_{\{0,\infty\}} : \mathbb{P}^1_k \to \mathbb{P}^1_k/(\mathbb{P}^1_k - \{0,\infty\})$, we will need to consider the pinch maps arising from $c_D : \mathbb{P}^1_k \to \mathbb{P}^1_k/(\mathbb{P}^1_k - D)$ for arbitrary divisors $D \subset \mathbb{P}^1_k(k)$. We begin by defining the algebraic pinch map associated to D.

Lemma 4.1. *Let $x \in \mathbb{P}^1_k(k)$ be a rational point. Then there exists a homotopy inverse $p : \frac{\mathbb{P}^1_k}{\mathbb{P}^1_k - \{x\}} \to \mathbb{P}^1_k$ to c_x in $\mathcal{H}_\bullet(k)$.*

Proof. The collapse map c_x is a homotopy equivalence by [Hoy14, Lemma 5.4], which implies the existence of a homotopy inverse p. In fact, an explicit formula for p is given in *loc. cit.* □

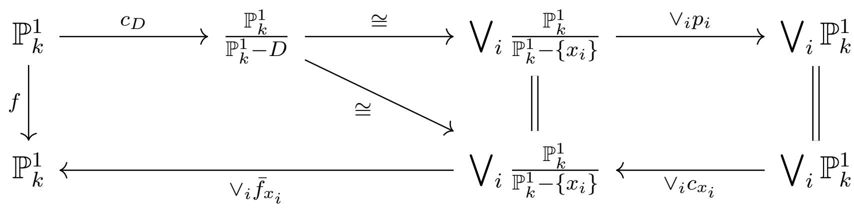
Definition 4.2. Let $D = \{x_1, \ldots, x_n\} \subset \mathbb{P}^1_k(k)$ be a finite set of rational points. Define the *D-pinch map* as the composite

$$\curlyvee_D : \mathbb{P}^1_k \xrightarrow{c_D} \frac{\mathbb{P}^1_k}{\mathbb{P}^1_k - D} \xrightarrow{\simeq} \bigvee_{i=1}^n \frac{\mathbb{P}^1_k}{\mathbb{P}^1_k - \{x_i\}} \xrightarrow{\vee_i p_i} \bigvee_{i=1}^n \mathbb{P}^1_k,$$

where c_D is the collapse map induced by the inclusion $\mathbb{P}^1_k - D \hookrightarrow \mathbb{P}^1_k$, the second map is the canonical isomorphism of motivic spaces $\mathbb{P}^1_k/(\mathbb{P}^1_k - D) \cong \bigvee_{i=1}^n \mathbb{P}^1_k/(\mathbb{P}^1_k - \{x_i\})$ given by [Caz12, Lemma A.3], and $p_i = c_{x_i}^{-1}$ (which exists by Lemma 4.1) for each i.

Homotopically, the desired local-to-global principle for the unstable degree should be encoded as the commutativity of the following diagram, which relates our "global" map $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ to an appropriate sum $\vee_i(\bar{f}_{x_i} \circ c_{x_i}) \circ \curlyvee_D : \mathbb{P}^1_k \to \mathbb{P}^1_k$ of its local terms.

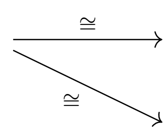
(4.3)



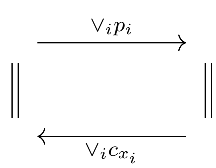
Theorem 4.3 (Local-to-global principle, homotopically)**.** *Let $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ be a pointed rational map with vanishing locus $D = \{x_1, \ldots, x_n\} \subset \mathbb{P}^1_k(k)$. Then $f = \vee_i(\bar{f}_{x_i} \circ c_{x_i}) \circ \curlyvee_D$ in $\mathcal{H}_\bullet(k)$.*

Proof. The top three maps of Diagram 4.3 compose to $\curlyvee_D$. Thus if Diagram 4.3 commutes in $\mathcal{H}_\bullet(k)$, then we obtain the desired result by comparing the leftmost vertical map with the composite around the remaining three edges of the outer rectangle.

There are three polygons in Diagram 4.3 to consider. The commutativity of the central triangle

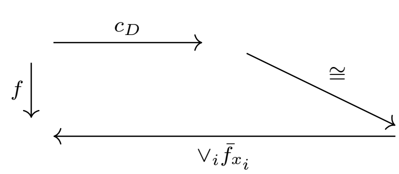


is simply two copies of the isomorphism $\frac{\mathbb{P}^1_k}{\mathbb{P}^1_k - D} \cong \bigvee_i \frac{\mathbb{P}^1_k}{\mathbb{P}^1_k - \{x_i\}}$ [Caz12, Lemma A.3]. The commutativity of the rightmost rectangle



follows from Lemma 4.1, which states that p_i is the homotopy inverse of c_{x_i} in $\mathcal{H}_\bullet(k)$.

Finally, we need to show that the leftmost trapezoid



commutes. The commutativity of this diagram is explained in Remark 3.2. $\square$

In summary, we have proved that a pointed rational function is homotopic to the sum of its homotopical local unstable degrees. The subtlety in this story is figuring out *which definition* of addition ensures this local-to-global principle. Theorem 4.3 states that taking our addition to be $(-)\circ\Upsilon_D$, where D is the vanishing locus of $f:\mathbb{P}^1_k\to\mathbb{P}^1_k$, gives us the desired local-to-global principle for f. This justifies the following definition.

Definition 4.4. Let $D=\{r_1,\ldots,r_n\}\subset\mathbb{A}^1_k(k)$. The *(homotopical) D-sum* is the function

$$\sum_D := (-)\circ\Upsilon_D : [\mathbb{P}^1_k,\mathbb{P}^1_k]^n \to [\mathbb{P}^1_k,\mathbb{P}^1_k].$$

If we do not wish to specify the divisor D, we may refer to the D-sum as a *(homotopical) divisorial sum.*

Our next goal is to study the algebraic image $\bigoplus_D := \deg^u\circ\sum_D$ of the D-sum and compare it to the usual group structure on $\mathrm{GW}^u(k)$.

5. Aside on duplicants

Before computing the addition law $\bigoplus_D$ in $\mathrm{GW}^u(k)$, we need to generalize the notion of the discriminant of a polynomial. We begin with some notation.

Notation 5.1. Given $m,n\in\mathbb{N}$, denote the m^{th} elementary symmetric polynomial in n variables by

$$\sigma_{m,n}(x_1,\ldots,x_n) := \sum_{1\leq i_1<\ldots<i_m\leq n} x_{i_1}\cdots x_{i_m}.$$

By convention, we will set $\sigma_{0,n}=1$ and $\sigma_{m,n}=0$ for $m\notin\{0,\ldots,n\}$.

Given a monic polynomial of the form $f=\prod_{i=1}^n(x-r_i)^{e_i}$, let $N:=\deg(f)$ and

$$\boldsymbol{r}_{i,j} := (\underbrace{r_1,\ldots,r_1}_{e_1\text{ times}},\ldots,\underbrace{r_i,\ldots,r_i}_{e_i-j\text{ times}},\ldots,\underbrace{r_n,\ldots,r_n}_{e_n\text{ times}}).$$

By Vieta's formulas, the coefficient of x^i in $f/(x-r_\ell)^j=(x-r_\ell)^{e_\ell-j}\prod_{m\neq\ell}(x-r_m)^{e_m}$ is given by $(-1)^{N-i-j}\sigma_{N-i-j,N-j}(\boldsymbol{r}_{\ell,j})$. For fixed ℓ and varying $0\leq i\leq N-1$ and $1\leq j\leq e_\ell$, we get a matrix of coefficients $\Sigma_\ell(f):=((-1)^{N-i-j}\sigma_{N-i-j,N-j}(\boldsymbol{r}_{\ell,j}))_{i,j}$. If we treat i as the row index and j as the column index, then the matrix

$$\Sigma(f) := \begin{pmatrix}\Sigma_1(f) & \Sigma_2(f) & \cdots & \Sigma_n(f)\end{pmatrix}$$

is an $N\times N$ square. We will only be interested in $\det\Sigma(f)$ and its square, so we will conflate $\Sigma(f)$ and its transpose $\Sigma(f)^\intercal$ when convenient.

The heavy notation needed for this setup is unfortunate, as it may obfuscate what $\Sigma(f)$ really is:

Proposition 5.2. *Let $f/g : \mathbb{P}^1_k \to \mathbb{P}^1_k$ be a pointed rational function. Assume that $f = \prod_{i=1}^n (x-r_i)^{e_i}$ with $N := \sum_{i=1}^n e_i$. Then the change-of-basis matrix from the monomial basis*

$$\left\{\frac{1}{g(x)}, \frac{x}{g(x)}, \ldots, \frac{x^{N-1}}{g(x)}\right\}$$

to the Newton basis

$$\left\{\frac{f(x)}{(x-r_1)g(x)}, \ldots, \frac{f(x)}{(x-r_1)^{e_1}g(x)}, \ldots, \frac{f(x)}{(x-r_n)g(x)}, \ldots, \frac{f(x)}{(x-r_n)^{e_n}g(x)}\right\}$$

is given by $\Sigma(f)^\intercal$.

Proof. By definition, $\Sigma_\ell(f)$ is the matrix of coefficients of $f/(x-r_\ell), \ldots, f/(x-r_\ell)^{e_\ell}$. This matrix is indexed so that

$$\Sigma_\ell(f)^\intercal \begin{pmatrix} \frac{1}{g(x)} \\ \vdots \\ \frac{x^{N-1}}{g(x)} \end{pmatrix} = \begin{pmatrix} \frac{f(x)}{(x-r_\ell)g(x)} \\ \vdots \\ \frac{f(x)}{(x-r_\ell)^{e_\ell}g(x)} \end{pmatrix}.$$

It follows that $\Sigma(f)^\intercal$ is the desired change-of-basis matrix. □

Remark 5.3. Note that the change-of-basis matrix in Proposition 5.2 does not depend on $g(x)$, justifying the notation $\Sigma(f)$.

We will need to work with $\det \Sigma(f)^2$ in Section 6, so we give it a name and derive a formula for it.

Definition 5.4. Let $f \in k[x]$ be a monic polynomial whose roots are all k-rational. Under the conventions listed in Notation 5.1, we define the *duplicant* of f as

$$\mathfrak{D}(f) := \det \Sigma(f)^2.$$

Example 5.5. Let $f = (x-r_1)(x-r_2)^2$. Then $\Sigma_1(f) = \begin{pmatrix} r_2^2 & -2r_2 & 1 \end{pmatrix}$ and

$$\Sigma_2(f) = \begin{pmatrix} r_1r_2 & -r_1-r_2 & 1 \\ -r_1 & 1 & 0 \end{pmatrix}.$$

Setting $f_{\mathrm{red}} = (x-r_1)(x-r_2)$, we compute

$$\begin{aligned} \mathfrak{D}(f) &= \det \begin{pmatrix} r_2^2 & -2r_2 & 1 \\ r_1r_2 & -r_1-r_2 & 1 \\ -r_1 & 1 & 0 \end{pmatrix}^2 \\ &= (r_1-r_2)^4 \\ &= \mathrm{disc}(f_{\mathrm{red}})^2. \end{aligned}$$

See Appendix A for some rough Sage code for computing duplicants.

The following proposition shows that the duplicant is indeed a generalization of the discriminant.

Proposition 5.6. *Let $f = \prod_{i=1}^{n}(x - r_i)$ with all r_i distinct. Then $\mathfrak{D}(f) = \operatorname{disc}(f)$.*

Proof. Since $e_i = 1$ for all i, the matrices of coefficients take the form

$$\Sigma_\ell(f) := ((-1)^{N-i-1}\sigma_{N-i-1,N-1}(r_1, \ldots, \hat{r}_\ell, \ldots, r_n))_{i=0}^{N-1}.$$

Note that

$$\begin{aligned}\frac{\partial \sigma_{a,b}}{\partial x_\ell} &= \sum_{1 \le i_1 < \ldots < \ell < \ldots < i_a \le b} x_{i_1} \cdots \hat{x}_\ell \cdots x_{i_a} \\ &= \sigma_{a-1,b-1}(x_1, \ldots, \hat{x}_\ell, \ldots, x_b)\end{aligned}$$

when $1 \le a \le b$. It follows that, up to multiplying some rows by -1, we have

$$\begin{aligned}\Sigma(f) &= \begin{pmatrix} \sigma_{n-1,n-1}(\boldsymbol{r}_{1,1}) & \sigma_{n-2,n-1}(\boldsymbol{r}_{1,1}) & \cdots & \sigma_{0,n-1}(\boldsymbol{r}_{1,1}) \\ \sigma_{n-1,n-1}(\boldsymbol{r}_{2,1}) & \sigma_{n-2,n-1}(\boldsymbol{r}_{2,1}) & \cdots & \sigma_{0,n-1}(\boldsymbol{r}_{2,1}) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n-1,n-1}(\boldsymbol{r}_{n,1}) & \sigma_{n-2,n-1}(\boldsymbol{r}_{n,1}) & \cdots & \sigma_{0,n-1}(\boldsymbol{r}_{n,1}) \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial \sigma_{n,n}}{\partial x_1} & \frac{\partial \sigma_{n-1,n}}{\partial x_1} & \cdots & \frac{\partial \sigma_{1,n}}{\partial x_1} \\ \frac{\partial \sigma_{n,n}}{\partial x_2} & \frac{\partial \sigma_{n-1,n}}{\partial x_2} & \cdots & \frac{\partial \sigma_{1,n}}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \sigma_{n,n}}{\partial x_n} & \frac{\partial \sigma_{n-1,n}}{\partial x_n} & \cdots & \frac{\partial \sigma_{1,n}}{\partial x_n} \end{pmatrix}\Bigg|_{x_i = r_i},\end{aligned}$$

where the evaluation sets $x_i = r_i$ for all $1 \le i \le n$. Thus

$$\det \Sigma(f) = \pm \operatorname{Jac}(\sigma_{n,n}, \ldots, \sigma_{1,n})|_{x_i = r_i}.$$

In order to compute $\det \Sigma(f)^2$, it therefore suffices to evaluate the Jacobian determinant of the elementary symmetric polynomials. The computation

$$\operatorname{Jac}(\sigma_{1,n}, \ldots, \sigma_{n,n}) = \prod_{1 \le i < j \le n} (x_i - x_j)$$

is classical (see e.g. [Per51, pp. 150]) and implies

$$\operatorname{Jac}(\sigma_{n,n}, \ldots, \sigma_{1,n}) = (-1)^{\lfloor n/2 \rfloor} \prod_{1 \le i < j \le n} (x_i - x_j).$$

After evaluating $x_i \mapsto r_i$, this squares to $\operatorname{disc}(f)$. □

Based on computations using the code in Appendix A, we can conjecture (and subsequently prove) a compact formula for $\mathfrak{D}(f)$.

Theorem 5.7. *If $f = \prod_{i=1}^{n}(x - r_i)^{e_i}$, then*

$$\det \Sigma(f) = \pm \prod_{1 \le i < j \le n} (r_i - r_j)^{e_i e_j},$$

and hence

$$\mathfrak{D}(f) = \prod_{1 \le i < j \le n} (r_i - r_j)^{2 e_i e_j}.$$

Proof. Let $N := \sum_{i=1}^n e_i$. Consider the monomial, slant monomial, and Newton bases of $Q(f) := k[x]/(f)$:

$$\begin{aligned}
B^{\text{mon}}(f) &= \Big\{1, x, x^2, \ldots, x^{N-1}\Big\},\\
B^{\text{slant}}(f) &= \Big\{1, (x-r_1), \ldots, (x-r_1)^{e_1},\\
&\quad (x-r_1)^{e_1}(x-r_2), \ldots, (x-r_1)^{e_1}(x-r_2)^{e_2},\\
&\quad \ldots,\\
&\quad \prod_{i=1}^{n-1}(x-r_i)^{e_i}\cdot(x-r_n), \ldots, \prod_{i=1}^{n-1}(x-r_i)^{e_i}\cdot(x-r_n)^{e_n-1}\Big\},\\
B^{\text{Nwt}}(f) &= \bigcup_{i=1}^n \Big\{\frac{f}{x-r_i}, \ldots, \frac{f}{(x-r_i)^{e_i}}\Big\}.
\end{aligned}$$

Given bases B and B', denote the B-to-B' change-of-basis matrix by $T^B_{B'}$. To simplify notation, we will write $T^{\text{mon}}_{\text{Nwt}}(f) := T^{B^{\text{mon}}(f)}_{B^{\text{Nwt}}(f)}$, and similarly for other pairs of bases among $B^{\text{mon}}(f), B^{\text{slant}}(f), B^{\text{Nwt}}(f)$. By Proposition 5.2, we can prove the present theorem by showing that $\det T^{\text{mon}}_{\text{Nwt}}(f) = \pm\prod_{i<j}(r_i - r_j)^{e_ie_j}$.

Note that $T^{\text{mon}}_{\text{slant}}(f)$ is a triangular matrix with all entries on the diagonal equal to 1, since the elements of $\text{mon}(f)$ and $\text{slant}(f)$ are monic polynomials of degrees $0, 1, \ldots, N-1$. In particular, $\det T^{\text{mon}}_{\text{slant}}(f) = 1$, so $\det T^{\text{mon}}_{\text{Nwt}}(f) = \det T^{\text{slant}}_{\text{Nwt}}(f)$. We will thus compute $\det T^{\text{slant}}_{\text{Nwt}}(f)$.

We conclude the proof by inducting on n. The base case is $n = 1$, in which $T^{\text{slant}}_{\text{Nwt}}(f)$ is a permutation matrix (and thus has determinant ± 1) and $\prod_{1\leq i<j\leq n}(r_i - r_j)^{e_ie_j}$ is an empty product (and thus equal to 1). As the inductive hypothesis, we may therefore assume

$$\det T^{\text{slant}}_{\text{Nwt}}(\tilde{f}) = \pm\prod_{1\leq i<j\leq n-1}(r_i - r_j)^{e_ie_j},$$

where $\tilde{f} = \prod_{i=1}^{n-1}(x-r_i)^{e_i}$ (so that $f = \tilde{f}\cdot(x-r_n)^{e_n}$). We will complete the inductive step in Lemma 7.23. □

Lemma 5.8. *Assume the notation of Theorem 5.7 and its proof. If* $\det T^{\text{slant}}_{\text{Nwt}}(\tilde{f}) = \pm\prod_{1\leq i<j\leq n-1}(r_i - r_j)^{e_ie_j}$, *then* $\det T^{\text{slant}}_{\text{Nwt}}(f) = \pm\prod_{1\leq i<j\leq n}(r_i - r_j)^{e_ie_j}$.

Proof. Note that

$$B^{\text{Nwt}}(f) = \Big\{v(x)\cdot(x-r_n)^{e_n} : v(x)\in B^{\text{Nwt}}(\tilde{f})\Big\} \cup \Big\{\frac{f}{x-r_n}, \ldots, \frac{f}{(x-r_n)^{e_n}}\Big\},$$

$$B^{\text{slant}}(f) = B^{\text{slant}}(\tilde{f}) \cup \Big\{\frac{f}{(x-r_n)^{e_n}}, \ldots, \frac{f}{x-r_n}\Big\}. \tag{5.1}$$

This implies that $T_{\mathrm{Nwt}}^{\mathrm{slant}}(f)$ is a block diagonal matrix: the rows of $T_{\mathrm{Nwt}}^{\mathrm{slant}}(f)$ corresponding to the $\{f/(x-r_n),\dots,f/(x-r_n)^{e_n}\}$ are 0 in the columns corresponding to $B^{\mathrm{slant}}(\tilde{f})$ and a permutation matrix in the remaining columns. Similarly, the rows of $T_{\mathrm{Nwt}}^{\mathrm{slant}}(f)$ corresponding to the elements $\{v(x)\cdot(x-r_n)^{e_n} : v(x)\in B^{\mathrm{Nwt}}(\tilde{f})\}$ are the first $\sum_{i=1}^{n-1} e_i$ rows of the product

$$M\cdot T_{\mathrm{Nwt}}^{\mathrm{slant}}(\tilde{f})$$

(followed by e_n columns of zeros), where M is the $N\times(\sum_{i=1}^{n-1} e_i)$ matrix corresponding to the linear transformation $Q(\tilde{f})\to Q(f)$ given by multiplication by $(x-r_n)^{e_n}$ on $B^{\mathrm{slant}}(\tilde{f})$.

By Equation 5.1, the first $\sum_{i=1}^{n-1} e_i$ rows of M correspond to the elements of $B^{\mathrm{slant}}(\tilde{f})$. In particular, the matrix M consists of a square matrix S with rows and columns indexed by $B^{\mathrm{slant}}(\tilde{f})$, followed by e_n rows underneath that are irrelevant for our computations. The matrix S can be written as $P\cdot M$, where P is the matrix of the projection $Q(f)\to Q(\tilde{f})$ corresponding to forgetting the basis elements $B^{\mathrm{slant}}(f)-B^{\mathrm{slant}}(\tilde{f})=\{f/(x-r_n)^{e_n},\dots,f/(x-r_n)\}$.

All of this setup allows us to state

$$\begin{aligned}\det T_{\mathrm{Nwt}}^{\mathrm{slant}}(f) &= \pm\det\left(M\cdot T_{\mathrm{Nwt}}^{\mathrm{slant}}(\tilde{f})\right)_{i,j=1}^{N-e_n}\\ &= \pm\det(P\cdot M)\cdot\det T_{\mathrm{Nwt}}^{\mathrm{slant}}(\tilde{f}).\end{aligned}$$

It thus suffices to prove that $\det(P\cdot M)=\prod_{i=1}^{n-1}(r_i-r_n)^{e_ie_n}$. Note that if we write

$$F=\prod_{i=1}^{n}\prod_{j=1}^{e_i}(x-r_{i,j})$$

and treat $r_{i,j}$ as variables, then $B^{\mathrm{slant}}(F)$ is a basis for the free $k[r_{1,1},\dots,r_{n,e_n}]$-module given by polynomials in $k[r_{1,1},\dots,r_{n,e_n}][x]$ of degree at most $N-1$. Similarly, writing

$$\tilde{F}=\prod_{i=1}^{n-1}\prod_{j=1}^{e_i}(x-r_{i,j}),$$

we have that $B^{\mathrm{slant}}(\tilde{F})$ is a basis for the free $k[r_{1,1},\dots,r_{n-1,e_{n-1}}]$-module given by polynomials in $k[r_{1,1},\dots,r_{n-1,e_{n-1}}][x]$ of degree at most $N-e_n-1$. Specializing $r_{i,j}\mapsto r_i$ sends $F\mapsto f$ and $\tilde{F}\mapsto\tilde{f}$. In particular, we can compute $\det(P\cdot M)$ by working with $B^{\mathrm{slant}}(F)$ and $B^{\mathrm{slant}}(\tilde{F})$ and then specializing. By inductively specializing, beginning with $r_{n,j}$ and working down to $r_{1,j}$, we may therefore assume that $e_i=1$ for $1\le i\le n-1$.

Now let

$$\begin{aligned} v_1 &= 1, \\ v_2 &= x - r_1, \\ v_3 &= (x - r_1)(x - r_2) \\ &\vdots \\ v_n &= (x - r_1) \cdots (x - r_{n-1}), \end{aligned}$$

so that $B^{\text{slant}}(f) = \{v_1, \ldots, v_n, \frac{f}{(x-r_n)^{e_n}}, \ldots, \frac{f}{x-r_n}\}$. We then define constants $a_{i,j} \in k$ by

$$v_i(x) \cdot (x - r_n)^{e_n} = \sum_{j=1}^{N-e_n} a_{i,j} \cdot v_j(x) + R_i(x), \tag{5.2}$$

where $R_i(x)$ is a k-linear combination of the basis elements $\{\frac{f}{(x-r_n)^{e_n}}, \ldots, \frac{f}{x-r_n}\}$. As matrices, we have

$$P \cdot M = (a_{i,j})_{i,j=1}^n,$$

so we need to show that $\det(a_{i,j}) = \prod_{i=1}^{n-1}(r_i - r_n)^{e_n}$ (recall that we have assumed $e_i = 1$ for $i < n$). Note that $R_i(r_\ell) = 0$ for all $0 \leq \ell < n$. Similarly, $v_i(r_\ell) = 0$ for $i > \ell$. Substituting $x = r_\ell$ into Equation 5.2 for $1 \leq \ell < n$, we find that

$$a_{i,j} = \begin{cases} (r_i - r_n)^{e_n} & i = j, \\ 0 & i < j. \end{cases}$$

This implies that $\det(P \cdot M) = \prod_{i=1}^{n-1}(r_i - r_n)^{e_n}$ when $e_1 = \ldots = e_{n-1} = 1$, which completes the proof. □

Remark 5.9. If we loosen the requirement that f be monic, we can still define and compute the duplicant of f. If $f \in k[x]$ with all roots $r_1, \ldots, r_n$ rational, then we can write $f = c \cdot h$, where $h = \prod_{i=1}^n (x - r_i)^{e_i}$ and $c \in k^\times$. The coefficient matrix $\Sigma(f)$ is now given by scaling each column of $\Sigma(h)$ by c, so we find that

$$\begin{aligned} \det \Sigma(f) &= c^{\operatorname{rank} \Sigma(h)} \cdot \det \Sigma(h) \\ &= c^{\sum_i e_i} \cdot \det \Sigma(h). \end{aligned}$$

If we define $\mathfrak{D}(f) := \det \Sigma(f)^2$ and denote $N := \deg(f) = \deg(h) = \sum_{i=1}^n e_i$, then it follows from Theorem 5.7 that

$$\mathfrak{D}(f) = c^{2N} \prod_{1 \leq i < j \leq n} (r_i - r_j)^{2e_i e_j}.$$

Unlike the usual discriminant, the duplicant need not vanish when f has repeated roots. In fact, since $\mathfrak{D}(f)$ is the square of the determinant of the monomial-to-Newton change-of-basis matrix, we have $\mathfrak{D}(f) \neq 0$.

6. Local-to-global principle, algebraically

Our next goal is to derive an algebraic formula for the homotopical D-sum given in Theorem 4.3. We will begin by showing that this sum must be more subtle than the natural group structure on $\mathrm{GW}^u(k)$. To do so, we need to recall Cazanave's monoid operation on $[\mathbb{P}^1_k, \mathbb{P}^1_k]$ (whose group completion maps under $\deg^u$ to the standard group structure on $\mathrm{GW}^u(k)$) [Caz12, §3.1].

Definition 6.1. Let f be a polynomial with $\deg(f) = n$. Then there is a unique pair of polynomials u, v with $\deg(u) \leq n-2$ and $\deg(v) \leq n-1$ satisfying the Bézout identity $fu + gv = 1$. Given two pointed rational functions f_1/g_1 and f_2/g_2, let u_i, v_i be the corresponding pairs of polynomials. Write

$$\begin{pmatrix} f_3 & -v_3 \\ g_3 & u_3 \end{pmatrix} := \begin{pmatrix} f_1 & -v_1 \\ g_1 & u_1 \end{pmatrix} \begin{pmatrix} f_2 & -v_2 \\ g_2 & u_2 \end{pmatrix}.$$

Then the *naïve sum* is defined to be $f_1/g_1 \oplus^{\mathrm{N}} f_2/g_2 := f_3/g_3$, which is again a pointed rational function.

By specifying the monoid structure on $[\mathbb{P}^1_k, \mathbb{P}^1_k]$ in Theorem 2.9, Cazanave effectively gives a local-to-global principle for computing the unstable degree in terms of $\mathrm{Béz}^{\mathrm{mon}}$. However, we will see that this naïve local-to-global principle does not satisfy our desired criteria. The shortcoming is that when decomposing a pointed rational function f/g by the naïve sum, the resulting "local" terms do not vanish at the same points as the original function f/g.

Instead, we will show that the local Newton matrix, namely our formula for the unstable local degree, satisfies a local-to-global principle with respect to the divisorial sum (see Definitions 4.4 and 6.4).

6.1. **Insufficiency of the naïve local-to-global principle.** By Theorem 2.9, one can express the unstable degree of a pointed rational function f/g as a sum of unstable degrees of rational functions $f_1/g_1, \ldots, f_n/g_n$ of lesser degree. Iterating this process decreases the degrees of the naïve summands, so one can assume that each f_i/g_i vanishes at a single point in $\mathbb{P}^1_k$. The unstable local degree of such a function should be equal to its unstable (global) degree, so this gives a *naïve* local-to-global principle for the unstable degree. Unfortunately, the point of vanishing of f_i/g_i can never belong to the vanishing locus of f/g:

Proposition 6.2. *Let $f/g : \mathbb{P}^1_k \to \mathbb{P}^1_k$ be a pointed rational function. Assume that $f = f_1 \cdot f_2$ for some non-constant polynomials f_1, f_2. Then there cannot exist g_1, g_2 such that $f_i/g_i : \mathbb{P}^1_k \to \mathbb{P}^1_k$ are pointed rational functions with $f/g = f_1/g_1 \oplus^{\mathrm{N}} f_2/g_2$.*

Proof. Suppose that $f/g = f_1/g_1 \oplus^{\mathrm{N}} f_2/g_2$ with $f = f_1 \cdot f_2$. By definition of $\oplus^{\mathrm{N}}$, we have $f = f_1 f_2 - v_1 g_2$, so $v_1 g_2 = 0$. Since g_2 is the denominator of a pointed rational function and the ring of polynomials over a field is a domain, we deduce that $v_1 = 0$. But this implies that $f_1 u_1 = 1$, so f_1 is a unit. It follows that f_1 must be constant, contradicting our assumption that f_1, f_2 are non-constant. □

Corollary 6.3. *Let $f/g : \mathbb{P}^1_k \to \mathbb{P}^1_k$ be a pointed rational function with vanishing locus $\{x_1, \ldots, x_n\}$. For each x_i, let m_i be its minimal polynomial. Let $e_i \cdot \deg(m_i)$ be the order of vanishing of f at x_i, so that $f = \prod_{i=1}^n m_i^{e_i}$. Then there cannot exist polynomials $g_1, \ldots, g_n$ such that $m_i^{e_i}/g_i : \mathbb{P}^1_k \to \mathbb{P}^1_k$ are pointed rational functions satisfying*

$$\deg^u(f/g) = \sum_{i=1}^{n} \deg^u(m_i^{e_i}/g_i). \tag{6.1}$$

Proof. By [Caz12, Theorem 3.6], finding $g_1, \ldots, g_n$ satisfying Equation 6.1 is equivalent to finding $g_1, \ldots, g_n$ such that

$$\frac{f}{g} = \frac{m_1^{e_1}}{g_1} \oplus^{\mathrm{N}} \cdots \oplus^{\mathrm{N}} \frac{m_n^{e_n}}{g_n}.$$

Since $\oplus^{\mathrm{N}}$ is associative, we can reduce via induction to the $n = 2$ case. It now follows from Proposition 6.2 that such a factorization cannot exist. □

Corollary 6.3 tells us that the Bézoutian with respect to the monomial basis will not give a satisfactory *unstable local* degree, in contrast with the unstable *global* degree [Caz12] and the *stable* local degree [BMP23]. This is because the local terms in any naïve decomposition will not vanish at any points in the vanishing locus of our original function.

6.2. **Divisorial sums of local terms.** We have just seen that in general, the naïve sum will not give us a satisfactory local-to-global principle. In Theorem 4.3, we saw that our desired local-to-global principle requires that we work with the homotopical sum $(-) \circ \curlyvee_D$, where D is the vanishing locus of the pointed rational map that we are trying to decompose. In contrast, the naïve sum is defined homotopically by collapsing the complement of the locus $\{0, \infty\}$. In other words, the naïve sum fails to give the desired local-to-global principle, because the vanishing locus of a pointed rational map is generally not a subset of $\{0, \infty\}$.

Our next goal is to compute the image in $\mathrm{GW}^u(k)$ (under the Bézoutian) of the addition law $\sum_D := (-) \circ \curlyvee_D$. We will also call this image the (algebraic) D-sum, denoted $\oplus_D$, which will depend on D. We will use Theorem 4.3, our formula for the unstable local degree, and Cazanave's formula for the unstable global degree to compute $\oplus_D$.

Definition 6.4. Let $D = \{r_1, \ldots, r_n\} \subset \mathbb{P}^1_k(k)$ be a finite set of rational points. The *(algebraic) D-sum* is the function

$$\bigoplus_D : \bigoplus_{i=1}^{n} \mathrm{GW}^u(k) \to \mathrm{GW}^u(k)$$

satisfying $\bigoplus_D \deg^u(f_i) = \deg^u(\vee_i f_i \circ \curlyvee_D)$ for any n-tuple $f_1, \ldots, f_n : \mathbb{P}^1_k \to \mathbb{P}^1_k$ of pointed rational maps. In other words, $\bigoplus_D := \deg^u \circ \sum_D$.

As with the homotopical D-sum, we will say *(algebraic) divisorial sum* when we do not wish to specify the divisor D.

Our goal is to give an algebraic formula for $\oplus_D$ for any n-tuple of elements in $\mathrm{GW}^u(k)$. As a first step, we can use the homotopical local-to-global principle (Theorem 4.3) and our formula for the unstable local degree (Lemma 3.10) to compute a formula for $\oplus_D$ in some cases.

Proposition 6.5. *Let $f/g : \mathbb{P}^1_k \to \mathbb{P}^1_k$ be a pointed rational function, with vanishing locus $D = \{r_1, \ldots, r_n\}$. Let $\deg^u_{r_i}(f/g) = (\beta_i, d_i) \in \mathrm{GW}^u(k)$, and let $m_i = \operatorname{rank} \beta_i$ and $m = \sum_i m_i$. Then*

$$\bigoplus_D ((\beta_1, d_1), \ldots, (\beta_n, d_n)) = \Big(\bigoplus_{i=1}^n \beta_i, \prod_{i=1}^n d_i \cdot \prod_{i<j} (r_i - r_j)^{2m_i m_j}\Big). \tag{6.2}$$

Proof. The unstable Grothendieck–Witt group $\mathrm{GW}^u(k)$ is the group completion of isomorphism classes of pairs $(\beta, b_1, \ldots, b_n)$ where β is a nondegenerate, symmetric bilinear form on a k-vector space with basis $b_1, \ldots, b_n$, and where an isomorphism is a linear isomorphism preserving the inner product and with determinant one in the given basis [Mor12, Remark 7.37]. We can therefore describe elements of $\mathrm{GW}^u(k)$ in terms of k-vector space, a choice of basis, and the Gram matrix of a symmetric bilinear form with respect to that basis.

Recall the notation $Q(f/g) := k[x, \frac{1}{g}]/(\frac{f}{g})$, and consider the following bases of $Q(f/g)$:

$$\begin{aligned}
B^{\mathrm{mon}}(f) &= \Big\{1, x, x^2, \ldots, x^{m-1}\Big\},\\
B^{\mathrm{mon}/g}(f) &= \Big\{\frac{1}{g}, \frac{x}{g}, \frac{x^2}{g}, \ldots, \frac{x^{m-1}}{g}\Big\},\\
B^{\mathrm{Nwt}}(f) &= \bigcup_{i=1}^n \Big\{\frac{f(x)}{(x-r_i)}, \frac{f(x)}{(x-r_i)^2}, \ldots, \frac{f(x)}{(x-r_i)^{m_i}}\Big\},\\
B^{\mathrm{Nwt}/g}(f) &= \bigcup_{i=1}^n \Big\{\frac{f(x)}{(x-r_i)g(x)}, \frac{f(x)}{(x-r_i)^2 g(x)}, \ldots, \frac{f(x)}{(x-r_i)^{m_i} g(x)}\Big\}.
\end{aligned}$$

By [Caz12, Theorem 3.6], the Gram matrix of $\deg^u(f/g)$ with respect to $B^{\mathrm{mon}}(f)$ is given by $\mathrm{Béz}^{\mathrm{mon}}(f/g) = (a_{ij})$, where

$$\frac{f(x)g(y) - f(y)g(x)}{x - y} = \sum_{i,j} a_{ij} x^i y^j. \tag{6.3}$$

Dividing both sides of Equation 6.3 by $g(x)g(y)$, we obtain

$$\frac{f(x)/g(x) - f(y)/g(y)}{x - y} = \sum_{i,j} a_{ij} \frac{x^i}{g(x)} \frac{y^j}{g(y)}.$$

As described in [KW20, Section 3 and Equation (22)], the (global) Newton matrix is given by $\mathrm{Nwt}(f/g) = (a_{ij})$, where

$$\frac{f(x)/g(x) - f(y)/g(y)}{x - y} = \sum_{i,j} a_{ij} v_i(x) v_j(y)$$

and $\{v_1, \ldots, v_{m-1}\} = B^{\mathrm{Nwt}/g}(f)$. This means that $\deg^u(f/g)$ and $\mathrm{Nwt}(f/g)$ are related by changing basis from $B^{\mathrm{mon}/g}(f)$ to $B^{\mathrm{Nwt}/g}(f)$. Note that the relevant change-of-basis matrix is identical to the change-of-basis matrix $T^{\mathrm{mon}}_{\mathrm{Nwt}}$ from $B^{\mathrm{mon}}(f)$ to $B^{\mathrm{Nwt}}(f)$. Moreover, $\mathrm{Nwt}(f/g) = \bigoplus_{i=1}^n \mathrm{Nwt}_{r_i}(f/g)$ by [KW20, Definition 7]. In summary, we find that

$$\begin{aligned}\deg^u(f/g) &= (T^{\mathrm{mon}}_{\mathrm{Nwt}})^\intercal \cdot \mathrm{Nwt}(f/g) \cdot T^{\mathrm{mon}}_{\mathrm{Nwt}} \\ &= (T^{\mathrm{mon}}_{\mathrm{Nwt}})^\intercal \cdot \bigoplus_{i=1}^n \mathrm{Nwt}_{r_i}(f/g) \cdot T^{\mathrm{mon}}_{\mathrm{Nwt}} \\ &= (T^{\mathrm{mon}}_{\mathrm{Nwt}})^\intercal \Big(\sum_{i=1}^n \deg^u_{r_i}(f/g) \Big) T^{\mathrm{mon}}_{\mathrm{Nwt}},\end{aligned}$$

where the last equality follows from Lemma 3.10. We conclude the proof by taking determinants and recalling that $\det(T^{\mathrm{mon}}_{\mathrm{Nwt}})^2 = \prod_{i<j}(r_i - r_j)^{2m_i m_j}$ by Theorem 5.7. □

Remark 6.6. Note that we can rewrite Equation 6.2 as

$$\bigoplus_D ((\beta_1, d_1), \ldots, (\beta_n, d_n)) = \Big(\bigoplus_{i=1}^n \beta_i, \mathfrak{D}(f) \cdot \prod_{i=1}^n d_i \Big),$$

where $\mathfrak{D}(f)$ is the duplicant of f.

7. Divisorial sums in general

In this section, we prove Theorem 1.1. Rather than working explicitly with $[\mathbb{P}^1_k, \mathbb{P}^1_k]$, we will work in terms of the fundamental group sheaf $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$. This is enabled by a theorem of Morel [Mor12, Remark 7.32], which is that there is a group isomorphism

$$[\mathbb{P}^1_k, \mathbb{P}^1_k] \cong \mathrm{End}(\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)(k)). \tag{7.1}$$

We think of Equation 7.1 as stating that $\mathbb{P}^1$ is $\mathbb{A}^1$-anabelian (as discussed in Section 1).

Using Equation 7.1 to translate to $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$, the general idea behind our proof of Theorem 1.1 is to show that the D-sum of $f_1, \ldots, f_n : \mathbb{P}^1 \to \mathbb{P}^1$ only depends on the free product $\ast_{i=1}^n \pi_1^{\mathbb{A}^1}(f_i)$ up to 2-nilpotence. We will need a fair amount of setup before we can employ this strategy.

Consider the Hopf map

$$\begin{aligned}\eta : \mathbb{A}^2 - \{0\} &\to \mathbb{P}^1 \\ (x, y) &\mapsto [x : y]\end{aligned}$$

and the map $\iota_{1,\infty} : \mathbb{P}^1 \to \mathbb{P}^\infty$ classifying $\mathcal{O}(1)$. Morel proved [Mor12, p. 191] that these together form an $\mathbb{A}^1$-fiber sequence

$$\mathbb{A}^2 - \{0\} \xrightarrow{\eta} \mathbb{P}^1 \xrightarrow{\iota_{1,\infty}} \mathbb{P}^\infty,$$

and that the induced long exact sequence of homotopy sheaves yields a central extension

$$1 \to \mathbf{K}_2^{\mathrm{MW}} \cong \pi_1^{\mathbb{A}^1}(\mathbb{A}^2 - \{0\}) \xrightarrow{\pi_1^{\mathbb{A}^1}(\eta)} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \xrightarrow{\pi_1^{\mathbb{A}^1}(\iota_{1,\infty})} \pi_1^{\mathbb{A}^1}(\mathbb{P}^\infty) \cong \mathbb{G}_m \to 1, \tag{7.2}$$

where $\mathbf{K}^{\mathrm{MW}}$ denotes the Milnor–Witt K-theory sheaf. Here, we use the following base points:

$$\begin{aligned}(1,0) &\in \mathbb{A}^2 - \{0\},\\ \eta(1,0) = [1:0] &\in \mathbb{P}^1,\\ \iota_{1,\infty}[1:0] = [1:0:\ldots] &\in \mathbb{P}^\infty.\end{aligned}$$

Moreover, we have a section

$$\theta : \mathbb{G}_m \to \Omega\Sigma\mathbb{G}_m \simeq \Omega\mathbb{P}^1$$

of $\wp$ by [Mor12, p. 191].

By Morel's anabelian theorem (Equation 7.1) and the isomorphism $[\mathbb{P}^1_k, \mathbb{P}^1_k] \cong \mathrm{GW}^u(k)$, we can associate to each $\beta \in \mathrm{GW}^u(k)$ and endomorphism $\pi_1^{\mathbb{A}^1}(\beta) : \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)(k) \to \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)(k)$. Let us use the notation $\wp := \pi_1^{\mathbb{A}^1}(\iota_{1,\infty}) : \pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \to \pi_1^{\mathbb{A}^1}(\mathbb{P}^\infty) \cong \mathbb{G}_m$. The composite of $\wp$ with (the image of) an arbitrary element in $\mathrm{GW}^u(k)$ depends only on the rank of that element:

Lemma 7.1. *Given $\beta \in \mathrm{GW}^u(k)$, let $\pi_1^{\mathbb{A}^1}(\beta)$ denote the corresponding element in $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$. If β has rank m, then*

$$\wp \circ \pi_1^{\mathbb{A}^1}(\beta) = m \circ \wp,$$

where $m : \mathbb{G}_m \to \mathbb{G}_m$ is given by $z \mapsto z^m$.

Proof. Given a sheaf of pointed sets $\mathcal{F}$, the *contraction* of $\mathcal{F}$ is the sheaf

$$\mathcal{F}_{-1} := \underline{\mathrm{Map}}(\mathbb{G}_m, \mathcal{F}).$$

(See e.g. [Mor12, Remark 2.23] or [Bac24, Section 4].) Contractions can be used to compute $[\mathbb{P}^1, \mathbb{P}^\infty] \cong \mathbb{Z}$. Indeed, the weak equivalence $\mathbb{P}^1 \simeq \Sigma\mathbb{G}_m$ implies that $[\mathbb{P}^1, \mathbb{P}^n] \cong [\Sigma\mathbb{G}_m, \mathbb{P}^n]$, and we have $[\Sigma\mathbb{G}_m, \mathbb{P}^n] \cong \pi_1^{\mathbb{A}^1}(\mathbb{P}^n)_{-1}(k)$ by definition of the contraction. Moreover, Morel proves that the $\mathbb{G}_m$-torsor $\mathbb{A}^{n+1} - \{0\} \to \mathbb{P}^n$ is the universal cover [Mor12, Theorem 7.13], which implies that $\pi_1^{\mathbb{A}^1}(\mathbb{P}^n) \cong \mathbb{G}_m$. It is now straightforward to compute the contraction $\pi_1^{\mathbb{A}^1}(\mathbb{P}^n)_{-1} \cong \mathbb{Z}$. Writing $[\mathbb{P}^1, \mathbb{P}^\infty]$ as a colimit, we then have

$$\begin{aligned}[\mathbb{P}^1, \mathbb{P}^\infty] &\cong \mathrm{colim}_n [\mathbb{P}^1, \mathbb{P}^n]\\ &\cong \mathrm{colim}_n\, \pi_1^{\mathbb{A}^1}(\mathbb{P}^n)_{-1}(k)\\ &\cong \mathbb{Z}.\end{aligned}$$

An explicit isomorphism $[\mathbb{P}^1, \mathbb{P}^\infty] \to \mathbb{Z}$ sends a map in $[\mathbb{P}^1, \mathbb{P}^\infty]$ to the degree of the corresponding pullback of $\mathcal{O}(1)$.

Recall that $\mathbb{P}^\infty \simeq \mathrm{B}\mathbb{G}_m$, so the map $m : \mathbb{G}_m \to \mathbb{G}_m$ induces a map $\mathrm{B}m : \mathbb{P}^\infty \to \mathbb{P}^\infty$. Both of the maps

$$\mathbb{P}^1 \xrightarrow{\beta} \mathbb{P}^1 \xrightarrow{\iota_{1,\infty}} \mathbb{P}^\infty,$$

$$\mathbb{P}^1 \xrightarrow{\iota_{1,\infty}} \mathbb{P}^\infty \xrightarrow{\mathrm{B}m} \mathbb{P}^\infty$$

classify $\mathcal{O}(m)$. By the above computation of $[\mathbb{P}^1, \mathbb{P}^\infty] \cong \mathbb{Z}$, this implies that $\mathrm{B}m \circ \iota_{1,\infty} \simeq \iota_{1,\infty} \circ \beta$. We conclude by noting that $\pi_1^{\mathbb{A}^1}(\mathrm{B}m \circ \iota_{1,\infty}) = m \circ \wp$ and $\pi_1^{\mathbb{A}^1}(\iota_{1,\infty} \circ \beta) = \wp \circ \pi_1^{\mathbb{A}^1}(\beta)$. □

Corollary 7.2. *If $\beta \in \mathrm{GW}^u(k)$ has rank 0, then we have a natural transformation*

$$\pi_1^{\mathbb{A}^1}(\beta) : \pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \to \mathbf{K}_2^{\mathrm{MW}}.$$

Proof. Lemma 7.1 implies that $\pi_1^{\mathbb{A}^1}(\beta)$ lies in the kernel of $\wp$. By Equation 7.2, the kernel of $\wp$ is the inclusion $\mathbf{K}_2^{\mathrm{MW}} \hookrightarrow \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$ via the Hopf map. Thus $\pi_1^{\mathbb{A}^1}(\beta) : \pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \to \mathbf{K}_2^{\mathrm{MW}}$, as desired. □

7.1. **Good pinch maps.** Our next step is to study the effect of $\pi_1^{\mathbb{A}^1}$ on maps of the form $\mathbb{P}^1 \to \bigvee_i \mathbb{P}^1$. We begin with a definition.

Definition 7.3. Given $n \in \mathbb{N}$ and $1 \leq j \leq n$, let

$$\iota_j : \mathbb{P}^1 \to \bigvee_{i=1}^n \mathbb{P}^1,$$
$$s_j : \bigvee_{i=1}^n \mathbb{P}^1 \to \mathbb{P}^1$$

denote the inclusion of the j^{th} summand and projection onto the j^{th} summand, respectively. We say that a map $c : \mathbb{P}^1 \to \bigvee_{i=1}^n \mathbb{P}^1$ is a *good pinch map* is $s_j \circ c = \mathrm{id}_{\mathbb{P}^1}$ for all $j \in \{1, \ldots, n\}$.

The two pinch maps we will need for Theorem 1.1 are both good pinch maps:

Example 7.4 (Simplicial pinch)**.** The weak equivalence $\mathbb{P}^1 \simeq \Sigma\mathbb{G}_m$ (coming from $\mathbb{P}^1 = \mathbb{A}^1 \cup_{\mathbb{G}_m} \mathbb{A}^1$) and the standard pinch map $S^1 \to S^1 \vee S^1$ induce a good pinch map $c_+ : \mathbb{P}^1 \to \mathbb{P}^1 \vee \mathbb{P}^1$. We refer to c_+ as the *simplicial pinch.*

Example 7.5 (Divisorial pinch)**.** Let $D = \{r_1, \ldots, r_n\} \subset \mathbb{A}_k^1(k)$. Let $\curlyvee_D : \mathbb{P}_k^1 \to \bigvee_{i=1}^n \mathbb{P}_k^1$ be the D-pinch map (Definition 4.2). Then $\curlyvee_D$ is a good pinch map.

The next few results are devoted to the effect of $\pi_1^{\mathbb{A}^1}$ on good pinch maps, such as compatibility with the group structure on $\mathbf{K}_2^{\mathrm{MW}}$ or 2-nilpotent behavior. For most of these results, we need the following definitions.

Definition 7.6. A sheaf of groups $\mathcal{G}$ is called *strongly $\mathbb{A}^1$-invariant* if

$$\mathrm{H}^i_{\mathrm{Nis}}(X, \mathcal{G}) \to \mathrm{H}^i_{\mathrm{Nis}}(X \times \mathbb{A}^1, \mathcal{G})$$

is a bijection for $i = 0, 1$.

Definition 7.7. Let $\mathcal{G}r$ and $\mathcal{G}r_{\mathbb{A}^1}$ denote the categories of sheaves of groups and strongly $\mathbb{A}^1$-invariant sheaves of groups, respectively. Let

$$a : \mathcal{G}r \to \mathcal{G}r_{\mathbb{A}^1}$$

denote the left adjoint to the inclusion $\mathcal{G}r_{\mathbb{A}^1} \subset \mathcal{G}r$ [Mor12, Remark 7.11]. Given a family $\{\mathcal{G}_i\}$ of strongly $\mathbb{A}^1$-invariant sheaves, we define their sum as

$$\overset{\mathbb{A}^1}{\underset{i}{\ast}} \mathcal{G}_i := a\big(\underset{i}{\ast}\, \mathcal{G}_i\big),$$

where $\ast$ denotes the free product. We will generally omit the superscript $^{\mathbb{A}^1}$ from this notation, so the reader should take the initial strongly $\mathbb{A}^1$-invariant sheaf whenever a free product appears.

Lemma 7.8. *Let $\beta, \beta_0 \in \mathrm{GW}^u(k)$ with $\operatorname{rank} \beta_0 = 0$. Let $c : \mathbb{P}^1 \to \mathbb{P}^1 \vee \mathbb{P}^1$ be a good pinch map. For all $U \in \mathrm{Sm}_k$ and $\gamma \in \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)(U)$, we have*

$$\pi_1^{\mathbb{A}^1}((\beta \vee \beta_0) \circ c)(\gamma) = \pi_1^{\mathbb{A}^1}(\beta)(\gamma) + \pi_1^{\mathbb{A}^1}(\beta_0)(\gamma).$$

Proof. First, we need to explain the notation $\pi_1^{\mathbb{A}^1}(\beta)(\gamma) + \pi_1^{\mathbb{A}^1}(\beta_0)(\gamma)$. Since $\mathbf{K}_2^{\mathrm{MW}}$ lies in the center of $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$ by Equation 7.2, there is an induced addition homomorphism

$$+ : \pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \times \mathbf{K}_2^{\mathrm{MW}} \to \pi_1^{\mathbb{A}^1}(\mathbb{P}^1).$$

We can therefore add $\pi_1^{\mathbb{A}^1}(\beta)(\gamma) \in \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)(U)$ to $\pi_1^{\mathbb{A}^1}(\beta_0)(\gamma) \in \mathbf{K}_2^{\mathrm{MW}}(U)$, using the natural transformation $\pi_1^{\mathbb{A}^1}(\beta_0) : \pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \to \mathbf{K}_2^{\mathrm{MW}}$ given by Corollary 7.2.

By Morel's Seifert–van Kampen theorem [Mor12, Theorem 7.12], the canonical maps $\pi_1^{\mathbb{A}^1}(\iota_i)$ assemble to an isomorphism

$$a\big(\overset{n}{\underset{i=1}{\ast}} \pi_1^{\mathbb{A}^1}(\iota_i)\big) : a\big(\overset{n}{\underset{i=1}{\ast}} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)\big) \xrightarrow{\cong} \pi_1^{\mathbb{A}^1}\big(\bigvee_{i=1}^{n} \mathbb{P}^1\big). \tag{7.3}$$

Under Equation 7.3, the map $\pi_1^{\mathbb{A}^1}(\beta \vee \beta_0) : \pi_1^{\mathbb{A}^1}(\mathbb{P}^1 \vee \mathbb{P}^1) \to \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$ is equivalent to the map

$$a(\pi_1^{\mathbb{A}^1}(\beta) \ast \pi_1^{\mathbb{A}^1}(\beta_0)) : a(\pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \ast \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)) \to \pi_1^{\mathbb{A}^1}(\mathbb{P}^1).$$

Recall that $\mathbb{A}^1$-fundamental group sheaves (over a field) are always strongly $\mathbb{A}^1$-invariant [Mor12, Theorem 1.9]. The same is true for Milnor–Witt K-theory sheaves by [Mor12, Theorem 3.37]. Thus $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \times \mathbf{K}_2^{\mathrm{MW}}$ is strongly $\mathbb{A}^1$-invariant (as cohomology factors over finite products of sheaves), so we have an equivalence $a(\pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \times \mathbf{K}_2^{\mathrm{MW}}) \cong \pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \times \mathbf{K}_2^{\mathrm{MW}}$. By Corollary 7.2 and since $\mathbf{K}_2^{\mathrm{MW}}$ lies in the center of $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$, the map $a(\pi_1^{\mathbb{A}^1}(\beta) \ast \pi_1^{\mathbb{A}^1}(\beta_0))$ factors through

$$\pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \times \mathbf{K}_2^{\mathrm{MW}} \xrightarrow{+} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1).$$

In total, we obtain a map

$$g : a(\pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \ast \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)) \to \pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \times \mathbf{K}_2^{\mathrm{MW}}$$

induced by $a(\pi_1^{\mathbb{A}^1}(\beta) * \pi_1^{\mathbb{A}^1}(\beta_0))$. It follows that the diagram

$$\begin{array}{ccc} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1 \vee \mathbb{P}^1) & \xrightarrow{\times_{i=1}^2 \pi_1^{\mathbb{A}^1}(s_i)} & \pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \times \pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \\ {\scriptstyle a(*_{i=1}^2 \pi_1^{\mathbb{A}^1}(\iota_i))}\uparrow & & \downarrow{\scriptstyle \pi_1^{\mathbb{A}^1}(\beta)\times\pi_1^{\mathbb{A}^1}(\beta_0)} \\ a(\pi_1^{\mathbb{A}^1}(\mathbb{P}^1) * \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)) & \xrightarrow{g} & \pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \times \mathbf{K}_2^{\mathrm{MW}} \end{array}$$

commutes. Since c is a good pinch map, the composite

$$\pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \xrightarrow{\pi_1^{\mathbb{A}^1}(c)} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1 \vee \mathbb{P}^1) \xrightarrow{\pi_1^{\mathbb{A}^1}(\beta\circ s_1)\times\pi_1^{\mathbb{A}^1}(\beta_0\circ s_2)} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \times \mathbf{K}_2^{\mathrm{MW}} \xrightarrow{+} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$$

is equivalent to the composite

$$\pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \xrightarrow{\pi_1^{\mathbb{A}^1}(\beta)\times\pi_1^{\mathbb{A}^1}(\beta_0)} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \times \mathbf{K}_2^{\mathrm{MW}} \xrightarrow{+} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1).$$

Thus $\pi_1^{\mathbb{A}^1}((\beta \vee \beta_0) \circ c) = \pi_1^{\mathbb{A}^1}(\beta) + \pi_1^{\mathbb{A}^1}(\beta_0)$, as desired. □

Corollary 7.9. *Let* $\beta, \beta_0 \in \mathrm{GW}^u(k)$ *with* $\operatorname{rank}\beta_0 = 0$. *For all* $U \in \mathrm{Sm}_k$ *and* $\gamma \in \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)(U)$, *we have*

$$\pi_1^{\mathbb{A}^1}(\beta + \beta_0)(\gamma) = \pi_1^{\mathbb{A}^1}(\beta)(\gamma) + \pi_1^{\mathbb{A}^1}(\beta_0)(\gamma).$$

Proof. This follows immediately from Lemma 7.8 by taking $c := c_+$ to be the simplicial pinch (Example 7.4), since $(\beta \vee \beta_0) \circ c_+ = \beta + \beta_0$ in $\mathrm{GW}^u(k)$ [Caz12, Lemma 3.20 and Theorem 3.21]. □

7.2. **2-nilpotence.** Our next step is to work up to 2-nilpotence, which is a simplification enabled by the 2-nilpotence of $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$. This will allow us to give a convenient presentation of $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$.

Recall that a group G is called *2-nilpotent* if there is a normal subgroup $G_1 \triangleleft G$ such that

$$1 =: G_0 \triangleleft G_1 \triangleleft G_2 := G$$

forms a central series. In other words, the commutator subgroup $[G, G] \leq G_1$ is abelian, as G_1 lies in the center of G. We repackage this information in the following remark, which we will apply later.

Remark 7.10. Let

$$1 \to K \to G \to A \to 1 \tag{7.4}$$

be a central extension of groups with K and A abelian. Central extensions of A by K are in bijection with $\mathrm{H}^2(A; K)$. Given $g_1, g_2 \in G$, let $[g_1, g_2] := g_1 g_2 g_1^{-1} g_2^{-1}$ denote the commutator. The commutator gives a homomorphism

$$\begin{aligned} f : A \otimes A &\to K \\ a_1 \otimes a_2 &\mapsto [g_1, g_2], \end{aligned}$$

where $g_i \mapsto a_i$ under the surjection $G \to A$. Moreover, f determines the extension (Equation 7.4) up to isomorphism. Indeed, f is alternating and bilinear and therefore determines the extension via the inclusion $\mathrm{Hom}(A \wedge A, K) \hookrightarrow \mathrm{H}^2(A; K)$.

For a group sheaf $\mathcal{G}$, let

$$\mathcal{G} \to \mathcal{G}^{\text{2-nil}}$$

be the initial map to a sheaf of 2-nilpotent groups. It turns out divisorial sums factor through a 2-nilpotent sheaf.

Proposition 7.11. *Let $c : \mathbb{P}^1 \to \bigvee_{i=1}^n \mathbb{P}^1$ be a good pinch map. Let $f_1, \dots, f_n : \mathbb{P}^1 \to \mathbb{P}^1$ be unstable pointed $\mathbb{A}^1$-homotopy classes of maps. Then*

$$\pi_1^{\mathbb{A}^1}(\vee_i f_i \circ c) : \pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \to \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$$

factors through $a((*_{i=1}^n \pi_1^{\mathbb{A}^1}(\mathbb{P}^1))^{\text{2-nil}})^{\text{2-nil}}$.

Proof. Note that $\pi_1^{\mathbb{A}^1}(\vee_i f_i) : \pi_1^{\mathbb{A}^1}(\bigvee_{i=1}^n \mathbb{P}^1) \to \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$. As mentioned in Equation 7.3, Morel's Seifert–van Kampen theorem implies that the map

$$\mathop{*}_{i=1}^{n} \pi_1^{\mathbb{A}^1}(\iota_i) : \mathop{*}_{i=1}^{n} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \to \pi_1^{\mathbb{A}^1}\Big(\bigvee_{i=1}^{n} \mathbb{P}^1\Big)$$

induces an isomorphism $a(*_{i=1}^n \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)) \xrightarrow{\cong} \pi_1^{\mathbb{A}^1}(\bigvee_{i=1}^n \mathbb{P}^1)$. Under this isomorphism, $\pi_1^{\mathbb{A}^1}(\vee_i f_i)$ is identified with the map

$$a\Big(\mathop{*}_{i=1}^{n} \pi_1^{\mathbb{A}^1}(f_i)\Big) : a\Big(\mathop{*}_{i=1}^{n} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)\Big) \to \pi_1^{\mathbb{A}^1}(\mathbb{P}^1).$$

Since $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$ is 2-nilpotent (by Equation **??**), $*_{i=1}^n \pi_1^{\mathbb{A}^1}(f_i)$ factors as

$$\mathop{*}_{i=1}^{n} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \to \Big(\mathop{*}_{i=1}^{n} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)\Big)^{\text{2-nil}} \to \pi_1^{\mathbb{A}^1}(\mathbb{P}^1).$$

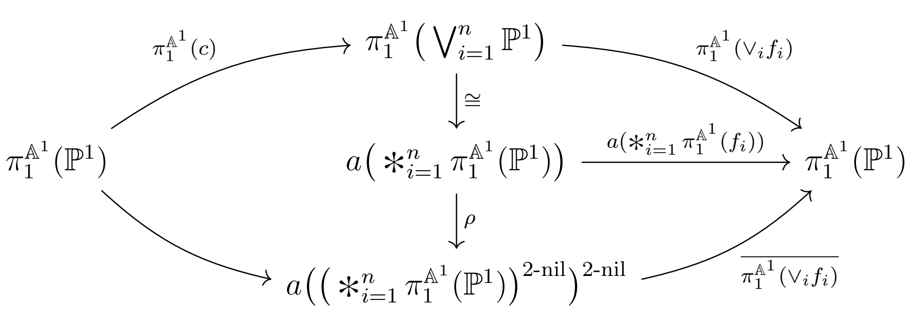
Since $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$ is strongly $\mathbb{A}^1$-invariant, we obtain a factorization

$$\mathop{*}_{i=1}^{n} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \to a\Big(\Big(\mathop{*}_{i=1}^{n} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)\Big)^{\text{2-nil}}\Big) \to \pi_1^{\mathbb{A}^1}(\mathbb{P}^1).$$

Finally, we again invoke the 2-nilpotence of $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$ to obtain a factorization

$$\mathop{*}_{i=1}^{n} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \to a\Big(\Big(\mathop{*}_{i=1}^{n} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)\Big)^{\text{2-nil}}\Big)^{\text{2-nil}} \to \pi_1^{\mathbb{A}^1}(\mathbb{P}^1).$$

Altogether, we get a commutative diagram



that yields the desired factorization. (See Notation 7.12 for the notation ρ and $\overline{\pi_1^{\mathbb{A}^1}(\vee_i f_i)}$, which we will need later.) $\square$

Notation 7.12. Let ρ denote the canonical map

$$\rho : a\Big(\mathop{*}_{i=1}^{n} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)\Big) \to a\Big(\Big(\mathop{*}_{i=1}^{n} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)\Big)^{\text{2-nil}}\Big)^{\text{2-nil}}.$$

Let

$$\overline{\pi_1^{\mathbb{A}^1}(\vee_i f_i)} : a\Big(\Big(\mathop{*}_{i=1}^{n} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)\Big)^{\text{2-nil}}\Big)^{\text{2-nil}} \to \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$$

be the unique map such that $\pi_1^{\mathbb{A}^1}(\vee_i f_i) = \overline{\pi_1^{\mathbb{A}^1}(\vee_i f_i)} \circ \rho$.

Our next goal is to give a nice presentation of $a((*_{i=1}^n \pi_1^{\mathbb{A}^1}(\mathbb{P}^1))^{\text{2-nil}})^{\text{2-nil}}$. Let $\mathcal{G}^{\text{ab}}$ denote the abelianization of a group sheaf $\mathcal{G}$. The following pushout diagram characterizes a simple case of free products with amalgamation for 2-nilpotent group sheaves.

Lemma 7.13. *Let $\mathcal{G}$ be a sheaf of 2-nilpotent groups. For each $j \in \{1, \ldots, n\}$, let $\iota_j : \mathcal{G}^{\text{ab}} \to *_{i=1}^n \mathcal{G}$ denote the inclusion of the j^{th} factor (given on group elements by amalgamating the identity element for each $i \neq j$). For $i \neq j$, let*

$$\begin{aligned}\varphi_{ij} : \mathcal{G}^{\text{ab}} \otimes \mathcal{G}^{\text{ab}} &\to \Big(\mathop{*}_{\ell=1}^{n} \mathcal{G}\Big)^{\text{2-nil}} \\ g_1 \otimes g_2 &\mapsto [\iota_i(g_1), \iota_j(g_2)].\end{aligned}$$

There is a pushout diagram

$$\begin{array}{ccc} \bigoplus_{i\neq j}(\mathcal{G}^{\text{ab}} \otimes \mathcal{G}^{\text{ab}}) & \xrightarrow{\bigoplus_{i\neq j}\varphi_{ij}} & \big(*_{i=1}^n \mathcal{G}\big)^{\text{2-nil}} \\ \downarrow & & \downarrow f \\ 1 & \longrightarrow & \times_{i=1}^n \mathcal{G} \end{array} \tag{7.5}$$

such that f is an epimorphism inducing a central extension.

Proof. Note that $*_{i=1}^n \mathcal{G} \to \times_{i=1}^n \mathcal{G}$ is an epimorphism, so

$$\Big(\mathop{*}_{i=1}^{n} \mathcal{G}\Big)^{\text{2-nil}} \to \Big(\mathop{\times}_{i=1}^{n} \mathcal{G}\Big)^{\text{2-nil}} \cong \mathop{\times}_{i=1}^{n} \mathcal{G}$$

is also an epimorphism. (Here, the last equivalence is because the product of 2-nilpotent groups is automatically 2-nilpotent.)

Next, the abelianization of $*_{i=1}^n \mathcal{G}$ factors through $\times_{i=1}^n \mathcal{G}$, so we have a factorization

$$\Big(\mathop{*}_{i=1}^{n} \mathcal{G}\Big)^{\text{2-nil}} \to \mathop{\times}_{i=1}^{n} \mathcal{G} \xrightarrow{\alpha} \Big(\mathop{*}_{i=1}^{n} \mathcal{G}\Big)^{\text{ab}}$$

with α an epimorphism. It follows that $(\ast_{i=1}^n \mathcal{G})^{\mathrm{ab}} \cong \times_{i=1}^n \mathcal{G}^{\mathrm{ab}}$, so the pushout diagram

$$\begin{array}{ccc} \bigoplus_{i\neq j}(\mathcal{G}^{\mathrm{ab}} \otimes \mathcal{G}^{\mathrm{ab}}) & \xrightarrow{\bigoplus_{i\neq j}\varphi_{ij}} & \left(\ast_{i=1}^n \mathcal{G}\right)^{2\text{-nil}} \\ \downarrow & & \downarrow f \\ 1 & \longrightarrow & \times_{i=1}^n \mathcal{G}^{\mathrm{ab}} \end{array} \tag{7.6}$$

implies that Diagram 7.5 is also a pushout. □

Since $\mathbf{K}_2^{\mathrm{MW}}$ and $\mathbb{G}_m$ are abelian group sheaves, the central extension given in Equation **??** implies that $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$ is a 2-nilpotent group sheaf (and hence so is $\times_{i=1}^n \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$). This means that the canonical map

$$\mathop{\ast}_{i=1}^{n} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \to \mathop{\times}_{i=1}^{n} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$$

factors through

$$\mathop{\ast}_{i=1}^{n} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \to \big(\mathop{\ast}_{i=1}^{n} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)\big)^{2\text{-nil}}.$$

As a finite product of strongly $\mathbb{A}^1$-invariant sheaves, the sheaf $\times_{i=1}^n \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$ is strongly $\mathbb{A}^1$-invariant, so applying the functors a and $(-)^{2\text{-nil}}$ to the above factorization gives us a map

$$\varphi : a\Big(\big(\mathop{\ast}_{i=1}^{n} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)\big)^{2\text{-nil}}\Big)^{2\text{-nil}} \to a\big(\mathop{\times}_{i=1}^{n} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)\big)^{2\text{-nil}} \cong \mathop{\times}_{i=1}^{n} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1). \tag{7.7}$$

We claim that φ is generated by commutators in the following sense:

Lemma 7.14. *Let* $\varphi : a((\ast_{i=1}^n \pi_1^{\mathbb{A}^1}(\mathbb{P}^1))^{2\text{-nil}})^{2\text{-nil}} \to \times_{i=1}^n \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$ *be the map constructed in Equation 7.7. Let* $K := \ker\varphi$.

(i) There is a surjection

$$\bigoplus_{i\neq j} a(\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)^{\mathrm{ab}} \otimes \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)^{\mathrm{ab}}) \to K$$

defined by summing (using the group law on K) the maps

$$a(\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)^{\mathrm{ab}} \otimes \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)^{\mathrm{ab}}) \to K$$

which are given over $U \in \mathrm{Sm}_k$ by

$$\begin{aligned} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)(U) \otimes \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)(U)) &\to K \\ \gamma_1 \otimes \gamma_2 &\mapsto [\pi_1^{\mathbb{A}^1}(\iota_i)(\gamma_1), \pi_1^{\mathbb{A}^1}(\iota_j)(\gamma_2)]. \end{aligned}$$

(ii) The map φ induces a central extension

$$1 \to K \to a\Big(\big(\mathop{\ast}_{i=1}^{n} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)\big)^{2\text{-nil}}\Big)^{2\text{-nil}} \xrightarrow{\varphi} \mathop{\times}_{i=1}^{n} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \to 1.$$

Proof. Since a is a left adjoint, a preserves epimorphisms and pushouts. We may thus apply a to Diagram 7.5 with $\mathcal{G} := \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$ to obtain a pushout of the form

$$\begin{array}{ccc} a\big(\bigoplus_{i\neq j}(\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)^{\mathrm{ab}} \otimes \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)^{\mathrm{ab}})\big) & \longrightarrow & a\big(\big(\ast_{i=1}^n \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)\big)^{2\text{-nil}}\big) \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & a\big(\times_{i=1}^n \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)\big). \end{array} \tag{7.8}$$

Let K' denote the image of the map

$$a\Big(\bigoplus_{i\neq j}\big(\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)^{\mathrm{ab}} \otimes \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)^{\mathrm{ab}}\big)\Big) \to a\Big(\Big(\mathop{\ast}_{i=1}^{n} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)\Big)^{2\text{-nil}}\Big),$$

which produces an extension

$$1 \to K' \to a\Big(\Big(\mathop{\ast}_{i=1}^{n} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)\Big)^{2\text{-nil}}\Big) \to a\Big(\mathop{\times}_{i=1}^{n} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)\Big) \cong \mathop{\times}_{i=1}^{n} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \to 1. \tag{7.9}$$

Here, the last isomorphism follows from the fact that $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$ (and hence $\times_{i=1}^n \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$) is strongly $\mathbb{A}^1$-invariant. Note that K' receives a surjection from $a\big(\bigoplus_{i\neq j}\big(\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)^{\mathrm{ab}} \otimes \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)^{\mathrm{ab}}\big)\big)$ by construction. Applying $(-)^{2\text{-nil}}$ (and recalling that $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$ and finite products thereof are 2-nilpotent), the extension given in Equation 7.9 surjects onto the extension

$$1 \to K \to a\Big(\Big(\mathop{\ast}_{i=1}^{n} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)\Big)^{2\text{-nil}}\Big)^{2\text{-nil}} \to \mathop{\times}_{i=1}^{n} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \to 1.$$

It follows that $a\big(\bigoplus_{i\neq j}\big(\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)^{\mathrm{ab}} \otimes \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)^{\mathrm{ab}}\big)\big) \to K$ is a surjection, as claimed.

For (ii), it remains to see that K is central in $a((\ast_{i=1}^n \pi_1^{\mathbb{A}^1}(\mathbb{P}^1))^{2\text{-nil}})^{2\text{-nil}}$. But commutators are always central in 2-nilpotent extensions, so the centrality of K follows from (i) and the 2-nilpotence of $a((\ast_{i=1}^n \pi_1^{\mathbb{A}^1}(\mathbb{P}^1))^{2\text{-nil}})^{2\text{-nil}}$. □

Next, we will describe how $\overline{\pi_1^{\mathbb{A}^1}(\vee_i f_i)}$ (see Notation 7.12) depends only on its value in $\mathbb{G}_m$ when restricted to the kernel K. To do so, we need to introduce more notation.

Notation 7.15. Since $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$ is a sheaf of 2-nilpotent groups, Remark 7.10 implies that there is an induced map

$$\mu : \mathbb{G}_m \otimes \mathbb{G}_m \to \mathbf{K}_2^{\mathrm{MW}} \tag{7.10}$$

which is defined on $U \in \mathrm{Sm}_k$ by

$$\begin{aligned} \mathbb{G}_m(U) \otimes \mathbb{G}_m(U) &\to \mathbf{K}_2^{\mathrm{MW}}(U) \\ \alpha_1 \otimes \alpha_2 &\mapsto [\theta(\alpha_1), \theta(\alpha_2)]. \end{aligned}$$

Here, $\theta : \mathbb{G}_m \to \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$ is the section of the central extension (Equation **??**) coming from the equivalence $\Omega\Sigma\mathbb{G}_m \simeq \Omega\mathbb{P}^1$.

Lemma 7.16. *Let* $f_1, \dots, f_n : \mathbb{P}^1_k \to \mathbb{P}^1_k$ *be endomorphisms of* $\mathbb{P}^1_k$*. Let* $m_i := \operatorname{rank} \deg^u(f_i)$ *for each* i*. Then the restriction of the morphism*

$$\overline{\pi_1^{\mathbb{A}^1}(\vee_i f_i)} : a\Big(\big(\mathop{\ast}_{i=1}^{n} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)\big)^{\text{2-nil}}\Big)^{\text{2-nil}} \to \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$$

to $K \leq a((\ast_{i=1}^n \pi_1^{\mathbb{A}^1}(\mathbb{P}^1))^{\text{2-nil}})^{\text{2-nil}}$ *fits in the commutative diagram*

$$\begin{array}{ccccc}
\bigoplus_{i\neq j} a(\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)^{\text{ab}} \otimes \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)^{\text{ab}}) & \twoheadrightarrow & K & \xrightarrow{\overline{\pi_1^{\mathbb{A}^1}(\vee_i f_i)}|_K} & \pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \\
\Big\downarrow{\scriptstyle \bigoplus_{i\neq j}(\wp\otimes\wp)} & & & & \Big\uparrow \\
\bigoplus_{i\neq j} \mathbb{G}_m \otimes \mathbb{G}_m & \xrightarrow{\bigoplus_{i\neq j}(m_i\otimes m_j)} & \bigoplus_{i\neq j} \mathbb{G}_m \otimes \mathbb{G}_m & \xrightarrow{\bigoplus_{i\neq j}\mu} & \mathbf{K}_2^{\text{MW}}.
\end{array}$$

Proof. We must show that the two maps

$$\bigoplus_{i\neq j} a(\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)^{\text{ab}} \otimes \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)^{\text{ab}}) \to \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$$

are equal. Since $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$ is strongly $\mathbb{A}^1$-invariant, it suffices to show that their precompositions with

$$\bigoplus_{i\neq j}(\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)^{\text{ab}} \otimes \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)^{\text{ab}}) \to \bigoplus_{i\neq j} a(\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)^{\text{ab}} \otimes \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)^{\text{ab}})$$

are equal. Fix $U \in \mathrm{Sm}_k$ and $p < q$. For $\gamma_1, \gamma_2 \in \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)(U)$, the tensor $\gamma_1 \otimes \gamma_2$ determines an element $(\gamma_1 \otimes \gamma_2)_{p,q}$ in the $(p,q)^{\text{th}}$ summand of $\bigoplus_{i\neq j}(\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)^{\text{ab}} \otimes \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)^{\text{ab}})$. Let $(\gamma_1 \otimes \gamma_2)_{p,q,K}$ denote the image of $(\gamma_1 \otimes \gamma_2)_{p,q}$ in K. Applying $\overline{\pi_1^{\mathbb{A}^1}(\vee_i f_i)}$, we compute

$$\begin{aligned}\overline{\pi_1^{\mathbb{A}^1}(\vee_i f_i)}(\gamma_1 \otimes \gamma_2)_{p,q,K} &= [\pi_1^{\mathbb{A}^1}(f_p)(\gamma_1), \pi_1^{\mathbb{A}^1}(f_q)(\gamma_2)] \\ &= \mu(\wp(\gamma_1)^{m_p}, \wp(\gamma_2)^{m_q}).\end{aligned}$$

This last equality follows from Lemma 7.1 and the existence of the map μ. □

7.3. **Proving Theorem 1.1.** We are almost ready to prove Theorem 1.1, which is an algebraic characterization of the D-sum in full generality. We need a few final lemmas.

Lemma 7.17. *Let* c_1 *and* c_2 *be good pinch maps. Then* $\Delta_{c_1,c_2} := (\rho \circ \pi_1^{\mathbb{A}^1}(c_1))(\rho \circ \pi_1^{\mathbb{A}^1}(c_2))^{-1}$ *determines a homomorphism of sheaves of groups*

$$\Delta_{c_1,c_2} : \pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \to K.$$

Proof. Let $U \in \mathrm{Sm}_k$. Mapping $\gamma \in \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)(U)$ to $\Delta_{c_1,c_2}(\gamma) := (\rho \circ \pi_1^{\mathbb{A}^1}(c_1))(\gamma)(\rho \circ \pi_1^{\mathbb{A}^1}(c_2))^{-1}(\gamma)$ determines a map of sheaves of sets

$$\pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \to a\Big(\big(\mathop{\ast}_{i=1}^{n} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)\big)^{\text{2-nil}}\Big)^{\text{2-nil}}.$$

Recall the map

$$\varphi : a\Big(\big(\mathop{\ast}_{i=1}^{n} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)\big)^{\text{2-nil}}\Big)^{\text{2-nil}} \to \mathop{\times}_{i=1}^{n} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$$

given in Equation 7.7. Since c_1 and c_2 are good pinch maps, we have

$$\varphi \circ \pi_1^{\mathbb{A}^1}(c_j) = (1,\ldots,1) : \pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \to \bigtimes_{i=1}^{n} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$$

for $j = 1, 2$. It follows that $\rho \circ \Delta_{c_1,c_2} = 0$, so Lemma 7.14 implies that Δ_{c_1,c_2} is in fact a map of sheaves of sets

$$\Delta_{c_1,c_2} : \pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \to K.$$

It remains to show that Δ_{c_1,c_2} is a homomorphism. For notational simplicity, let $g := \rho \circ \pi_1^{\mathbb{A}^1}(c_1)$ and $h = \rho \circ \pi_1^{\mathbb{A}^1}(c_2)$, so that $\Delta_{c_1,c_2} = gh^{-1}$. Note that ρ and $\pi_1^{\mathbb{A}^1}(c)$ for any good pinch map c are homomorphisms of group sheaves, so their composite is as well. Thus for any $\gamma_1, \gamma_2 \in \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)(U)$, we have that

$$\begin{aligned} g(\gamma_1\gamma_2)h(\gamma_1\gamma_2)^{-1} &= g(\gamma_1)g(\gamma_2) \cdot h(\gamma_2)^{-1}h(\gamma_1)^{-1} \\ &= g(\gamma_1)h(\gamma_1)^{-1} \cdot g(\gamma_2)h(\gamma_2)^{-1}. \end{aligned}$$

For the last equality, we use the fact that $K(U)$ is central in $a((\ast_{i=1}^n \pi_1^{\mathbb{A}^1}(\mathbb{P}^1))^{\text{2-nil}})^{\text{2-nil}}$ and that $gh^{-1}(\gamma_2) \in K(U)$. Thus Δ_{c_1,c_2} is a homomorphism as claimed. □

Recall that $\curlyvee_D$ and c_+ are good pinch maps. We will now relate $\overline{\pi_1^{\mathbb{A}^1}(\vee_i f_i)} \circ \Delta_{\curlyvee_D, c_+}$ to the divisorial sum of $f_1, \ldots, f_n$.

Notation 7.18. Let $D = \{r_1, \ldots, r_n\} \subset \mathbb{A}^1_k(k)$. Given elements $(\beta_1, d_1), \ldots, (\beta_n, d_n) \in \mathrm{GW}^u(k)$ with $m_i := \operatorname{rank} \beta_i$ for each i, define

$$\bigoplus_{D^{\mathrm{alg}}}((\beta_1, d_1), \ldots, (\beta_n, d_n)) := \Big(\bigoplus_{i=1}^n \beta_i, \prod_{i=1}^n d_i \cdot \prod_{i<j}(r_i - r_j)^{2m_im_j}\Big) \in \mathrm{GW}^u(k).$$

Given unstable pointed $\mathbb{A}^1$-homotopy classes of maps $f_1, \ldots, f_n : \mathbb{P}^1_k \to \mathbb{P}^1_k$, let

$$\bigoplus_D(\deg^u(f_1), \ldots, \deg^u(f_n)) := \deg^u\Big(\sum_D(f_1, \ldots, f_n)\Big).$$

To simplify notation, we will often write

$$\begin{aligned} \bigoplus_{D^{\mathrm{alg}}}(f_i) &:= \bigoplus_{D^{\mathrm{alg}}}(\deg^u(f_1), \ldots, \deg^u(f_n)), \\ \bigoplus_D(f_i) &:= \bigoplus_D(\deg^u(f_1), \ldots, \deg^u(f_n)). \end{aligned}$$

Note that Theorem 1.1 is equivalent to the statement $\bigoplus_D(f_i) = \bigoplus_{D^{\mathrm{alg}}}(f_i)$.

For integers $m_1, \ldots, m_n$, let

$$\beta_{m_1,\ldots,m_n,D} := \Big(0, \prod_{i<j}(r_i - r_j)^{2m_im_j}\Big) \in \mathrm{GW}^u(k).$$

Lemma 7.19. *Let $f_1, \ldots, f_n \in [\mathbb{P}^1_k, \mathbb{P}^1_k]$ with $m_i := \operatorname{rank}\deg^u(f_i)$ for each i. Then we have an equality*

$$(\overline{\pi_1^{\mathbb{A}^1}(\vee_i f_i)} \circ \Delta_{\curlyvee_D, c_+})\pi_1^{\mathbb{A}^1}(\beta_{m_1,\ldots,m_n,D}) = \pi_1^{\mathbb{A}^1}\big(\bigoplus_D (f_i)\big)\pi_1^{\mathbb{A}^1}\big(\bigoplus_{D^{\mathrm{alg}}}(f_i)\big)^{-1} \tag{7.11}$$

as maps of sheaves of sets $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \to \mathbf{K}_2^{\mathrm{MW}}$. Moreover, both of these maps are homomorphisms of sheaves of groups.

Proof. Consider the good pinch maps $c_1 = \curlyvee_D$ (Example 7.5) and $c_2 = c_+$ (Example 7.4). By definition, we have

$$\pi_1^{\mathbb{A}^1}\big(\bigoplus_D (f_i)\big) = \pi_1^{\mathbb{A}^1}((\vee_i f_i) \circ c_1). \tag{7.12}$$

By Corollary 7.2 and the fact that the simplicial pinch gives the group law on $\mathrm{GW}^u(k)$ [Caz12, Lemma 3.20 and Theorem 3.21], we have

$$\pi_1^{\mathbb{A}^1}\big(\bigoplus_{D^{\mathrm{alg}}} (f_i)\big) = \pi_1^{\mathbb{A}^1}((\vee_i f_i) \circ c_2)\pi_1^{\mathbb{A}^1}(\beta_{m_1,\ldots,m_n,D}), \tag{7.13}$$

with $\pi_1^{\mathbb{A}^1}(\beta_{m_1,\ldots,m_n,D})$ lying in the image of $\mathbf{K}_2^{\mathrm{MW}}$. Using the factorization proven in Proposition 7.11, we now have

$$\begin{aligned}
\pi_1^{\mathbb{A}^1}\big(\bigoplus_D (f_i)\big) &= \overline{\pi_1^{\mathbb{A}^1}(\vee_i f_i)} \circ (\rho \circ \pi_1^{\mathbb{A}^1}(c_1)),\\
\pi_1^{\mathbb{A}^1}\big(\bigoplus_{D^{\mathrm{alg}}} (f_i)\big) &= (\overline{\pi_1^{\mathbb{A}^1}(\vee_i f_i)} \circ (\rho \circ \pi_1^{\mathbb{A}^1}(c_2)))\pi_1^{\mathbb{A}^1}(\beta_{m_1,\ldots,m_n,D}).
\end{aligned}$$

The difference of these equalities is

$$\pi_1^{\mathbb{A}^1}\big(\bigoplus_D (f_i)\big)\pi_1^{\mathbb{A}^1}\big(\bigoplus_{D^{\mathrm{alg}}}(f_i)\big)^{-1} = (\overline{\pi_1^{\mathbb{A}^1}(\vee_i f_i)} \circ \Delta_{c_1,c_2})\pi_1^{\mathbb{A}^1}(\beta_{m_1,\ldots,m_n,D}),$$

giving the claimed equality of maps of sheaves of sets. By Lemmas 7.8 and 7.17, the map

$$(\overline{\pi_1^{\mathbb{A}^1}(\vee_i f_i)} \circ \Delta_{c_1,c_2})\pi_1^{\mathbb{A}^1}(\beta_{m_1,\ldots,m_n,D})$$

is a homomorphism of sheaves of groups, completing the proof. $\square$

Our next lemma states that if $\bigoplus_D (f_i) = \bigoplus_{D^{\mathrm{alg}}}(f_i)$ for some $f_1, \ldots, f_n$, then the same is true for any $f'_1, \ldots, f'_n$ satisfying $\operatorname{rank}\deg^u(f_i) = \operatorname{rank}\deg^u(f'_i)$. This is what will enable us to use Theorem 1.2 as the base case of an induction argument.

Lemma 7.20. *Let $D = \{r_1, \ldots, r_n\} \subset \mathbb{A}^1_k(k)$. Suppose we have two collections of endomorphisms*

$$\begin{aligned}
&f_1, \ldots, f_n \in [\mathbb{P}^1_k, \mathbb{P}^1_k],\\
&f'_1, \ldots, f'_n \in [\mathbb{P}^1_k, \mathbb{P}^1_k]
\end{aligned}$$

such that $\operatorname{rank}\deg(f_i) = \operatorname{rank}\deg(f_i')$ *for each* i*. If*

$$\bigoplus_D(\deg^u(f_1),\ldots,\deg^u(f_n)) = \bigoplus_{D^{\mathrm{alg}}}(\deg^u(f_1),\ldots,\deg^u(f_n)),$$

then we have

$$\bigoplus_D(\deg^u(f_1'),\ldots,\deg^u(f_n')) = \bigoplus_{D^{\mathrm{alg}}}(\deg^u(f_1'),\ldots,\deg^u(f_n')).$$

Proof. By Morel's anabelian theorem (Equation 7.1), it is enough to show that

$$\pi_1^{\mathbb{A}^1}(\bigoplus_D(f_i') = \pi_1^{\mathbb{A}^1}(\bigoplus_{D^{\mathrm{alg}}}(f_i').$$

Let $c_1 = \curlyvee_D$ and $c_2 = c_+$. By Lemma 7.19, we have

$$(\overline{\pi_1^{\mathbb{A}^1}(\vee_i f_i)} \circ \Delta_{c_1,c_2})\pi_1^{\mathbb{A}^1}(\beta_{m_1,\ldots,m_n,D}) = \pi_1^{\mathbb{A}^1}\big(\bigoplus_D(f_i)\big)\pi_1^{\mathbb{A}^1}\big(\bigoplus_{D^{\mathrm{alg}}}(f_i)\big)^{-1},$$

$$(\overline{\pi_1^{\mathbb{A}^1}(\vee_i f_i')} \circ \Delta_{c_1,c_2})\pi_1^{\mathbb{A}^1}(\beta_{m_1,\ldots,m_n,D}) = \pi_1^{\mathbb{A}^1}\big(\bigoplus_D(f_i')\big)\pi_1^{\mathbb{A}^1}\big(\bigoplus_{D^{\mathrm{alg}}}(f_i')\big)^{-1}.$$

Now Lemmas 7.16 and 7.17 imply that $(\overline{\pi_1^{\mathbb{A}^1}(\vee_i f_i)}\circ\Delta_{c_1,c_2})\pi_1^{\mathbb{A}^1}(\beta_{m_1,\ldots,m_n,D})$ only depends on $\operatorname{rank}\deg^u(f_1),\ldots,\operatorname{rank}\deg^u(f_n)$, and analogously for $(\overline{\pi_1^{\mathbb{A}^1}(\vee_i f_i)} \circ \Delta_{c_1,c_2})\pi_1^{\mathbb{A}^1}(\beta_{m_1,\ldots,m_n,D})$. In particular, our assumption that $\operatorname{rank}\deg^u(f_i) = \operatorname{rank}\deg^u(f_i')$ implies that

$$(\overline{\pi_1^{\mathbb{A}^1}(\vee_i f_i)} \circ \Delta_{c_1,c_2})\pi_1^{\mathbb{A}^1}(\beta_{m_1,\ldots,m_n,D}) = (\overline{\pi_1^{\mathbb{A}^1}(\vee_i f_i)} \circ \Delta_{c_1,c_2})\pi_1^{\mathbb{A}^1}(\beta_{m_1,\ldots,m_n,D})$$

Since $\bigoplus_D(f_i) = \bigoplus_{D^{\mathrm{alg}}}(f_i)$ by assumption, it follows that

$$(\overline{\pi_1^{\mathbb{A}^1}(\vee_i f_i)} \circ \Delta_{c_1,c_2})\pi_1^{\mathbb{A}^1}(\beta_{m_1,\ldots,m_n,D}) = 0,$$

namely the trivial map to the identity element in $\mathbf{K}_2^{\mathrm{MW}}$. We have thus proven that

$$\pi_1^{\mathbb{A}^1}\big(\bigoplus_D(f_i')\big)\pi_1^{\mathbb{A}^1}\big(\bigoplus_{D^{\mathrm{alg}}}(f_i')\big)^{-1},$$

so $\pi_1^{\mathbb{A}^1}(\bigoplus_D(f_i')) = \pi_1^{\mathbb{A}^1}(\bigoplus_{D^{\mathrm{alg}}}(f_i'))$. □

Corollary 7.21. *Let* $D = \{r_1,\ldots,r_n\} \subset \mathbb{A}_k^1(k)$*. Let*

$$f_1,\ldots,f_n \in [\mathbb{P}_k^1,\mathbb{P}_k^1]$$

satisfy $\operatorname{rank}\deg(f_i) > 0$ *for all* i*. Then*

$$\bigoplus_D(\deg^u(f_1),\ldots,\deg^u(f_n)) = \bigoplus_{D^{\mathrm{alg}}}(\deg^u(f_1),\ldots,\deg^u(f_n)).$$

Proof. Let $m_i := \operatorname{rank}\deg(f_i)$ for each i. By Lemma 7.20 and Proposition 6.5, it suffices to construct a pointed rational map $f : \mathbb{P}_k^1 \to \mathbb{P}_k^1$ with vanishing locus D such that $\operatorname{rank}\deg_{r_i}^u(f) = m_i$ for each i. The map $f := \prod_{i=1}^n (x - r_i)^{m_i}$ satisfies these criteria, e.g. by Lemma 3.10. □

We now come to the lemma embodying the inductive step of our proof of Theorem 1.1. The idea is to increment the rank of one of $\deg^u(f_1), \ldots, \deg^u(f_n)$.

Notation 7.22. Let δ_{ij} denote the Kronecker delta, defined by

$$\delta_{ij} = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

Lemma 7.23. *Let c_1 and c_2 be good pinch maps. Fix $\ell \in \{1, \ldots, n\}$. Suppose we have two collections of endomorphisms*

$$f_1, \ldots, f_n \in [\mathbb{P}^1_k, \mathbb{P}^1_k],$$
$$f'_1, \ldots, f'_n \in [\mathbb{P}^1_k, \mathbb{P}^1_k]$$

such that $\operatorname{rank} \deg^u(f_i) = \operatorname{rank} \deg^u(f'_i)$ *for each $i \neq \ell$. By abuse of notation, we write $f_\ell + \langle 1 \rangle^u$ to denote an endomorphism with unstable degree* $\deg^u(f_\ell) + \langle 1 \rangle^u$*, and similarly for $f'_\ell + \langle 1 \rangle^u$. Then we have an equality*

$$\overline{\pi_1^{\mathbb{A}^1}(\vee_i (f_i + \delta_{i\ell}\langle 1 \rangle^u))} \circ \Delta_{c_1,c_2} - \overline{\pi_1^{\mathbb{A}^1}(\vee_i f_i)} \circ \Delta_{c_1,c_2} =$$
$$\overline{\pi_1^{\mathbb{A}^1}(\vee_i (f'_i + \delta_{i\ell}\langle 1 \rangle^u))} \circ \Delta_{c_1,c_2} - \overline{\pi_1^{\mathbb{A}^1}(\vee_i f'_i)} \circ \Delta_{c_1,c_2}$$

are equal as maps $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \to \mathbf{K}_2^{\mathrm{MW}}$.

Proof. As before, let $m : \mathbb{G}_m \to \mathbb{G}_m$ denote the map $z \mapsto z^m$ for each integer m. For each pair of integers m_i, m_j, we can then form the maps

$$m_i \otimes m_j : \mathbb{G}_m \otimes \mathbb{G}_m \to \mathbb{G}_m \otimes \mathbb{G}_m.$$

Note that we have an equality $(m_i + 1) \otimes m_j = m_i \otimes m_j + 1 \otimes m_j$ of such maps, and hence an equality

$$\begin{aligned} \bigoplus_{i \neq j} (m_i + \delta_{i\ell}) \otimes m_j &= \bigoplus_{i \neq j} m_i \otimes m_j + \bigoplus_{i \neq j} \delta_{i\ell} \otimes m_j \\ &= \bigoplus_{i \neq j} m_i \otimes m_j + \bigoplus_{j \neq \ell} 1 \otimes m_j \end{aligned}$$

of maps

$$\bigoplus_{i \neq j} \mathbb{G}_m \otimes \mathbb{G}_m \to \bigoplus_{i \neq j} \mathbb{G}_m \otimes \mathbb{G}_m.$$

Now let $m_i := \operatorname{rank} \deg^u(f_i)$ for each i. We will show that

$$\overline{\pi_1^{\mathbb{A}^1}(\vee_i (f_i + \delta_{i\ell}\langle 1 \rangle^u))} \circ \Delta_{c_1,c_2} - \overline{\pi_1^{\mathbb{A}^1}(\vee_i f_i)} \circ \Delta_{c_1,c_2}$$

only depends on $m_1, \ldots, m_{\ell-1}, m_{\ell+1}, \ldots, m_n$. Our arguments will imply the same result when replacing the f_i with the f'_i, which will give the desired result.

Let $\sigma : \bigoplus_{i\neq j} a(\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)^{\mathrm{ab}} \otimes \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)^{\mathrm{ab}}) \to K$ denote the epimorphism of Lemma 7.14. Using Lemma 7.16, we compute that $\overline{\pi_1^{\mathbb{A}^1}(\vee_i(f_i + \delta_{i\ell}\langle 1\rangle^u))} \circ \sigma$ is equal to

$$\overline{\pi_1^{\mathbb{A}^1}(\vee_i f_i)} \circ \sigma + \Big(\bigoplus_{i\neq j} \mu\Big) \circ \Big(\bigoplus_{j\neq \ell} 1 \otimes m_j\Big) \circ \Big(\bigoplus_{j\neq\ell} \wp \otimes \wp\Big). \tag{7.14}$$

Note that

$$\Big(\bigoplus_{i\neq j} \mu\Big) \circ \Big(\bigoplus_{j\neq \ell} 1 \otimes m_j\Big) \circ \Big(\bigoplus_{j\neq\ell} \wp \otimes \wp\Big)$$

only depends on $m_1, \ldots, m_{\ell-1}, m_{\ell+1}, \ldots, m_n$ and is thus independent of m_ℓ. By Lemma 7.17, we have a homomorphism

$$\Delta_{c_1,c_2} : \pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \to K.$$

Together with Equation 7.14 and the fact that σ is an epimorphism, it follows that

$$\overline{\pi_1^{\mathbb{A}^1}(\vee_i(f_i + \delta_{i\ell}\langle 1\rangle^u))} \circ \Delta_{c_1,c_2} - \overline{\pi_1^{\mathbb{A}^1}(\vee_i f_i)} \circ \Delta_{c_1,c_2}$$

only depends on $m_1, \ldots, m_{\ell-1}, m_{\ell+1}, \ldots, m_n$. □

We can now put everything together and prove Theorem 1.1.

Proof of Theorem 1.1. Our goal is to show that for any $f_1, \ldots, f_n \in [\mathbb{P}^1_k, \mathbb{P}^1_k]$, we have $\bigoplus_D(f_i) = \bigoplus_{D^{\mathrm{alg}}}(f_i)$. By Morel's anabelian theorem (Equation 7.1), this is equivalent to proving that $\pi_1^{\mathbb{A}^1}(\bigoplus_D(f_i)) = \pi_1^{\mathbb{A}^1}(\bigoplus_{D^{\mathrm{alg}}}(f_i))$, which is what we will achieve after setting up our induction argument.

By Corollary 7.21, we have $\bigoplus_D(f_i) = \bigoplus_{D^{\mathrm{alg}}}(f_i)$ whenever each $\deg^u(f_i)$ has positive rank; this is the base case of our induction. For the inductive hypothesis, assume that we have $(m_1, \ldots, m_n) \in \mathbb{Z}^n$ such that $\bigoplus_D(f_i) = \bigoplus_{D^{\mathrm{alg}}}(f_i)$ whenever $\operatorname{rank}\deg^u(f_i) \geq m_i$ for each i.

We wish to show that if $f_1, \ldots, f_n \in [\mathbb{P}^1_k, \mathbb{P}^1_k]$ satisfy $\operatorname{rank}\deg^u(f_\ell) = m_\ell - 1$ for some $1 \leq \ell \leq n$ and $\operatorname{rank}\deg^u(f_i) = m_i$ for $i \neq \ell$, then $\bigoplus_D(f) = \bigoplus_{D^{\mathrm{alg}}}(f)$. Let $g_\ell \in [\mathbb{P}^1_k, \mathbb{P}^1_k]$ be an endomorphism such that $\deg^u(g_\ell) = \deg^u(f_\ell) + \langle 1\rangle^u$, which exists since $\mathrm{GW}^u(k) \cong [\mathbb{P}^1_k, \mathbb{P}^1_k]$. Let $g_i := f_i$ for all $i \neq \ell$. By our inductive hypothesis, we have

$$\bigoplus_D(g_i) = \bigoplus_{D^{\mathrm{alg}}}(g_i). \tag{7.15}$$

Thus Lemma 7.19 gives us

$$(\overline{\pi_1^{\mathbb{A}^1}(\vee_i g_i)} \circ \Delta_{\curlyvee_D, c_+})\pi_1^{\mathbb{A}^1}(\beta_{m_1,\ldots,m_\ell+1,\ldots,m_n,D}) = 0. \tag{7.16}$$

Similarly, we have

$$(\overline{\pi_1^{\mathbb{A}^1}(\vee_i(g_i + \delta_{i\ell}))} \circ \Delta_{\curlyvee_D, c_+})\pi_1^{\mathbb{A}^1}(\beta_{m_1,\ldots,m_\ell+2,\ldots,m_n,D}) = 0. \tag{7.17}$$

Now Lemma 7.23 implies that

$$\overline{\pi_1^{\mathbb{A}^1}(\vee_i(f_i + \delta_{i\ell}\langle 1\rangle^u))} \circ \Delta_{\curlyvee_D, c_+} - \overline{\pi_1^{\mathbb{A}^1}(\vee_i f_i)} \circ \Delta_{\curlyvee_D, c_+} =$$
$$\overline{\pi_1^{\mathbb{A}^1}(\vee_i(g_i + \delta_{i\ell}\langle 1\rangle^u))} \circ \Delta_{\curlyvee_D, c_+} - \overline{\pi_1^{\mathbb{A}^1}(\vee_i g_i)} \circ \Delta_{\curlyvee_D, c_+}.$$

Equation 7.16, Equation 7.17, and Lemma 7.8 give us

$$\begin{aligned}\overline{\pi_1^{\mathbb{A}^1}(\vee_i(g_i+\delta_{i\ell}\langle 1\rangle^u))}\circ\Delta_{\curlyvee_D,c_+}-\overline{\pi_1^{\mathbb{A}^1}(\vee_i g_i)}\circ\Delta_{\curlyvee_D,c_+}=\\ -\pi_1^{\mathbb{A}^1}(\beta_{m_1,\dots,m_\ell+2,\dots,m_n,D})+\pi_1^{\mathbb{A}^1}(\beta_{m_1,\dots,m_\ell+1,\dots,m_n,D})=\\ \pi_1^{\mathbb{A}^1}(-\beta_{m_1,\dots,m_\ell+2,\dots,m_n,D}+\beta_{m_1,\dots,m_\ell+1,\dots,m_n,D}).\end{aligned}$$

We can now directly calculate the unstable part of $-\beta_{m_1,\dots,m_\ell+2,\dots,m_n,D}+\beta_{m_1,\dots,m_\ell+1,\dots,m_n,D}$, which is

$$\prod_{i\neq j}(r_i-r_j)^{2(m_i+\delta_{i\ell})m_j}\cdot\prod_{i\neq j}(r_i-r_j)^{-2(m_i+2\delta_{i\ell})m_j}=\prod_{j\neq\ell}(r_\ell-r_j)^{-2m_j}.$$

In particular, we have

$$\overline{\pi_1^{\mathbb{A}^1}(\vee_i(f_i'+\delta_{i\ell}\langle 1\rangle^u))}\circ\Delta_{\curlyvee_D,c_+}-\overline{\pi_1^{\mathbb{A}^1}(\vee_i f_i')}\circ\Delta_{\curlyvee_D,c_+}=\pi_1^{\mathbb{A}^1}\Big(\big(0,\prod_{j\neq\ell}(r_\ell-r_j)^{-2m_j}\big)\Big).\tag{7.18}$$

Since $f_i+\delta_{i\ell}\langle 1\rangle^u=g_i$ for all i, Equations 7.16 and 7.18 imply that

$$\begin{aligned}\overline{\pi_1^{\mathbb{A}^1}(\vee_i f_i)}\circ\Delta_{\curlyvee_D,c_+}&=-\pi_1^{\mathbb{A}^1}(\beta_{m_1,\dots,m_{\ell+1},\dots,m_n,D})-\pi_1^{\mathbb{A}^1}\Big(\big(0,\prod_{j\neq\ell}(r_\ell-r_j)^{-2m_j}\big)\Big)\\&=-\pi_1^{\mathbb{A}^1}(\beta_{m_1,\dots,m_\ell,\dots,m_n,D}),\end{aligned}$$

where the last equality follows from Lemma 7.8. Thus

$$(\overline{\pi_1^{\mathbb{A}^1}(\vee_i f_i)}\circ\Delta_{\curlyvee_D,c_+})\pi_1^{\mathbb{A}^1}(\beta_{m_1,\dots,m_n,D})=0,$$

so $\pi_1^{\mathbb{A}^1}(\bigoplus_D(f_i))=\pi_1^{\mathbb{A}^1}(\bigoplus_{D^{\mathrm{alg}}}(f_i))$ by Lemma 7.19. □

Appendix A. Code

Here is some code that calculates duplicants and the square root of the ordinary discriminant. We used this code to conjecture a closed formula for the duplicant, which we then proved in Theorem 5.7.

```
def coefficients(f,N):
    # deal with constant term
    coeffs = [f.subs(x=0)]
    # append other coefficients
    for i in range(1,N):
        coeffs.append(f.coefficient(x^i))
    return(coeffs)

def vand(n): # compute sqrt(disc(r0,...,r(n-1)))
    r = var('r',n=n)
    factors = []
    for i in range(n):
        for j in range(i+1,n):
            factors.append(r[i]-r[j])
    return(prod(factors))
```

```
def dupl(n,e): # compute duplicant
    x = var('x')
    r = var('r',n=n)
    N = sum(e)
    coeff_list = []

    for i in range(n):
        for j in range(1,e[i]+1):
            # generate f/(x-r_i)^j
            e_new = e.copy()
            e_new[i] = e[i]-j
            f = prod([(x-r[l])^e_new[l]\
                      for l in range(n)]).expand()
            coeff_list.append(coefficients(f,N))

    # compute det^2 of coefficient matrix
    coeff_matrix = matrix(coeff_list)
    return(coeff_matrix.det()^2)
```

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