Heights over finitely generated fields

Stephen McKean
Duke University

Soumya Sankar
The Ohio State University

Abstract: This is an expository account about height functions and Arakelov theory in arithmetic geometry. We recall Conrad's description of generalized global fields in order to describe heights over function fields of higher transcendence degree. We then give a brief overview of Arakelov theory and arithmetic intersection theory. Our exposition culminates in a description of Moriwaki's Arakelov-theoretic formulation of heights, as well as a comparison of Moriwaki's construction to various versions of heights.

1 Introduction

A central goal in arithmetic geometry is to measure and compare the arithmetic complexity of points on an algebraic variety. For example, $[0 : 1]$ and $[49 : 54]$ are both rational points of the projective line, but the latter point is "more complicated" in a tractable way. The theory of heights provides such measures of complexity in the form of real-valued functions. Studying points of bounded height is of great interest from the point of view of arithmetic statistics and arithmetic geometry. Some such areas of extensive work include Manin's conjecture [1, 2, 3], Vojta's conjecture [4], and Bogomolov's conjecture (proved by Ullmo [5] and Zhang [6]). The theory of heights has also proved to be a powerful tool in arithmetic and algebraic geometry — many classical finiteness theorems, such as the Mordell–Weil theorem [7] and Faltings's theorem [8, 9], rely heavily on heights. See also Chambert-Loir's surveys on the subject [10, 11].

The formulation of geometric versions of the above conjectures and results (for instance the Lang–Néron theorem [12] or the geometric

Bogomolov conjecture [13]) requires making sense of heights over fields of arbitrary transcendence degree. At first pass, one usually constructs height functions on projective varieties over global fields. The set of valuations on a global field gives convenient real-valued functions, and the product formula enables one to fit these valuations together to obtain a well-defined height function. In this article, we discuss some heights over fields of higher transcendence degree. There are several possible approaches to heights in this case (see for instance, [14], or recent work of Yuan and Zhang [15] which generalizes Moriwaki's height discussed in this article). We choose two perspectives. First, following Conrad [12, Section 8], we give an introduction to *generalized global fields* in Section 2. The structure of these fields includes a set of valuations satisfying an appropriate generalization of the product formula, thus making it possible to construct a naive height over these fields in a similar manner to the case of global fields. We give a brief survey of naïve and geometric height functions in Section 3. See [16] for a classical discussion of these topics.

Second, we discuss Moriwaki's height ([13]) over finitely generated extensions of $\mathbf{Q}$. Moriwaki's definition of height is based on Arakelov theory and arithmetic intersection theory. A desirable property of height functions is that they reflect the geometry of the underlying variety in some sense. This leads to a "geometric" definition of height in terms of the degree of a line bundle. Classically, the geometric approach to height functions was used for varieties over function fields of curves. This was generalized to number fields using the work of Arakelov in [17]. In Part B of [18], Hindry and Silverman give a detailed exposition of the number field-function field analogy in the context of heights. In [13], Moriwaki generalized geometric heights to projective varieties over finitely generated extensions of $\mathbf{Q}$, using arithmetic intersection theory developed by Gillet and Soulé ([19]). The second half of this article is an exposition of this height function defined by Moriwaki. After reviewing the necessary ideas from Arakelov theory and arithmetic intersection theory in Section 5, we discuss Moriwaki's height in Section 6. We also discuss how Moriwaki's height recovers more familiar height functions over global fields [13]. In Theorem 1.67, we show that Moriwaki's height is induced by a generalized global field structure in certain cases.

We will assume that the reader is comfortable with line bundles on algebraic varieties. Some basic familiarity with valuation theory (see [20]) and intersection theory (see [21]) is also recommended.

Definition 1.1. A *valuation* on a field K is a map $v : K \to \Gamma \cup \{\infty\}$, where Γ is a totally ordered abelian group (we will simply take $\Gamma = \mathbf{R}$) such that (i) $v^{-1}(\infty) = 0$; (ii) $v(ab) = v(a) + v(b)$; and (iii) $v(a+b) \geq \min\{v(a), v(b)\}$, with equality if $a \neq b$. Two valuations are *equivalent* if they differ by an order-preserving group isomorphism on the target. A *place* on K is an equivalence class of valuations on K.

Notation 1.2. A *number field* is a finite extension of $\mathbf{Q}$. Given a number field K, we denote its *ring of integers* by $\mathscr{O}_K$. Similarly, we denote the *structure sheaf* of a scheme X by $\mathscr{O}_X$. A *global field* is either a number field or the function field of a curve over a finite field. Given a global field K, the *set of places* of K will be denoted M_K.

2 Generalized global fields

In Section 3, we will discuss various ways to define height functions on algebraic varieties over a field K. If K is a global field, we can use the theory of valuations to define heights on varieties over K. The key property of global fields that allows us to define a height in terms of valuations on K is the *product formula*. It turns out that if a field K satisfies a more general version of the product formula, we can still define heights on varieties over K in terms of valuations on K. This leads us to the notion of *generalized global fields*. We follow [12, Section 8] for our discussion of generalized global fields.

Notation 1.3. Given a field K with a valuation v, denote the completion of $(K, |\cdot|_v)$ by K_v. If v is Archimedean and $K_v \cong \mathbf{C}$, let $e_v := 2$. Let $e_v := 1$ in all other cases.

Definition 1.4. A *generalized global field* is a field K with infinitely many non-trivial places v and a choice of absolute value $|\cdot|_v$ for each v such that

(i) all but finitely many v are non-Archimedean,
(ii) each non-Archimedean v is discretely valued,
(iii) K_v/K is a separable extension for all non-Archimedean v,
(iv) for each $x \in K^\times$, we have $v(x) = 0$ for all but finitely many v, and
(v) for each $x \in K^\times$, the *generalized product formula* holds:

$$\prod_{\text{places } v} |x|_v^{e_v} = 1. \tag{1}$$

In order to show that the term "generalized" is justified, we need to check that global fields are examples of generalized global fields.

Proposition 1.5. *Every number field is a generalized global field.*

Proof Let K be a number field. Let M_K be the set of places of K. Given $v \in M_K$, let κ_v be the residue field of v if v is finite. The global field structure for K is given by the absolute values $\|\cdot\|_v$, where

$$\|\cdot\|_v = \begin{cases} |\kappa_v|^{-\operatorname{ord}_v(\cdot)} & v \text{ finite}, \\ |\cdot| & v \text{ infinite}. \end{cases}$$

Galois theory tells us that $[K : \mathbf{Q}]$ is the number of $\mathbf{Q}$-linear embeddings of K into $\overline{\mathbf{Q}} \subset \mathbf{C}$. Each of the infinitely many prime ideals $\mathfrak{p} \subset \mathscr{O}_K$ defines a distinct, discrete, non-Archimedean valuation. This verifies conditions (i) and (ii). Since $\operatorname{char} K = 0$, any extension of K is separable. Further, any element of K is a product of finitely many prime ideals in $\mathscr{O}_K$, so its $\mathfrak{p}$-adic valuation is 0 for all but finitely many finite places.

Finally, we need to show that the generalized product formula holds. Since the usual product formula holds with respect to the absolute values $\{\|\cdot\|_v\}$, the generalized product formula holds with respect to the absolute values $\{\|\cdot\|_v^{1/e_v}\}$. Thus the set $\{\|\cdot\|_v^{1/e_v}\}$ gives K the structure of a generalized global field. □

Global fields in positive characteristic do not have Archimedean places, so we do not need to check any of the Archimedean criteria for generalized global fields. Instead, we need to check that the completion of a global field K is a separable extension of K, which was not a concern in characteristic 0.

Proposition 1.6. *Every global field of positive characteristic is a generalized global field.*

Proof Let K be a finite extension of $\mathbf{F}_p(t)$ for some prime p. Each maximal ideal of the ring of integers of K determines a place of K, and these places are distinct if they are determined by distinct maximal ideals. Since there are infinitely many irreducible polynomials over $\mathbf{F}_p$ and K is a finite extension of $\mathbf{F}_p(t)$, there are infinitely many places of K. The absolute value corresponding to a maximal ideal $\mathfrak{m}$ is $\|\cdot\|_{\mathfrak{m}} = |\kappa_{\mathfrak{m}}|^{-\operatorname{ord}_{\mathfrak{m}}(\cdot)}$. Criteria (ii), (iv) and (v) are satisfied by these absolute values. In fact, (v) corresponds to the fact that a rational function on a complete, irreducible curve has degree 0.

It remains to show that K_v/K is a separable extension for all places of

K. Since K is a finite extension of $\mathbf{F}_p(t)$, [22, Theorem 1] implies that $K \cong \mathbf{F}_p(t, \alpha)$, where $\alpha \in K$ is a root of some irreducible polynomial in $\mathbf{F}_p(t)[x]$. At any place v, [23, II.4, Theorem 2] implies that $K_v \cong \mathbf{F}_{p^m}((u))$, where m is a positive integer and $u \in \mathcal{O}_v$ is a uniformizer. Since $\mathcal{O}_v \subseteq \mathbf{F}_p[t^{\pm 1}, \alpha^{\pm 1}]$, we can express u as a rational function in t and α over $\mathbf{F}_p$. In particular, K is an intermediate field of the extension $\mathbf{F}_{p^m}((u))/\mathbf{F}_p(u)$.

It thus suffices to prove that $\mathbf{F}_{p^m}((u))/\mathbf{F}_p(u)$ is separable. Since $\mathbf{F}_{p^m}/\mathbf{F}_p$ is separable, we just need to prove that $\mathbf{F}_s((u))/\mathbf{F}_s(u)$ is separable, where $s = p^m$. By [24, Lemma 2.6.1 (b)], it suffices to prove that if $f_1, \ldots, f_r \in \mathbf{F}_s((u))$ are linearly independent over $\mathbf{F}_s(u)$, then $f_1^p, \ldots, f_r^p$ are as well.

Suppose $\sum_{i=1}^r g_i f_i^p = 0$ for some $g_1, \ldots, g_r \in \mathbf{F}_s(u)$. By clearing denominators, we may assume that $f_1^p, \ldots, f_r^p \in \mathbf{F}_s[[u]]$ and $g_1, \ldots, g_r \in \mathbf{F}_s[u]$. We then write $g_i = \sum_{j=0}^{p-1} g_{ij}(u^p)u^j$, where each $g_{ij} \in \mathbf{F}_s[u]$. Since $x \mapsto x^p$ is an automorphism of $\mathbf{F}_s$, it follows that we may write $g_{ij}(u^p) = h_{ij}^p$ for some $h_{ij} \in \mathbf{F}_s[u]$. We now have

$$0 = \sum_{i=1}^r g_i f_i^p = \sum_{i=1}^r \left(\sum_{j=0}^{p-1} h_{ij}^p u^j \right) f_i^p = \sum_{j=0}^{p-1} \left(\sum_{i=1}^r h_{ij}^p f_i^p \right) u^j.$$

Note that $\sum_{i=1}^r h_{ij}^p f_i^p \in \mathbf{F}_s[[u^p]]$ (by the freshman's dream) and $j < p$ for all j, so the terms of $\sum_{i=1}^r g_i f_i^p$ of degree j mod p all belong to the $(\sum_{i=1}^r h_{ij}^p f_i^p)u^j$ summand. Since the degree j mod p terms of $\sum_{i=1}^r g_i f_i^p$ must sum to 0, we conclude that $\sum_{i=1}^r h_{ij}^p f_i^p = 0$ for all j. Thus $\sum_{i=1}^r h_{ij}^p f_i^p = (\sum_{i=1}^r h_{ij} f_i)^p = 0$ for all j, so $\sum_{i=1}^r h_{ij} f_i = 0$ for all j. By the $\mathbf{F}_s(u)$-linear independence of $f_1, \ldots, f_r$, it follows that $h_{ij} = 0$ for all i, j. Thus $g_i = 0$ for all i, so $f_1^p, \ldots, f_r^p$ are linearly independent over $\mathbf{F}_s(u)$. □

2.1 General function fields

While function fields of transcendence degree 1 are global fields, we would also like to describe heights associated to function fields of higher dimensional varieties. This is the main motivation behind generalized global fields: function fields of transcendence degree at least 2 are generalized global fields that are not global fields.

Example 1.7. (See also [25].) Let K/k be a finitely generated field extension with k algebraically closed in K. Assume $\operatorname{trdeg}(K/k) > 0$.

We will describe a generalized global field structure on K. For a concrete example, one can take $K = k(t_1, \dots, t_n)$.

Let V be a normal, integral, projective k-scheme such that $k(V) = K$. If $\operatorname{trdeg}(K/k) = 1$, then there is a unique such V. For each codimension 1 point $v \in V$, the order of vanishing of a rational function along v induces a valuation $\operatorname{ord}_v : K \to \mathbf{Z} \cup \{\infty\}$ with valuation ring $\mathscr{O}_{V,v}$. The valuation ord_v can be recovered from the valuation ring $\mathscr{O}_{V,v}$ (see e.g. [26, Tag 00I8]). The valuation rings $\mathscr{O}_{V,v}$, and hence the valuations ord_v, depend on the choice of model V (which is not unique when $\operatorname{trdeg}(K/k) > 1$), so we consider the model V of the extension K/k to be part of the generalized global field structure in this case.

We now check that K/k satisfies the criteria listed in Definition 1.4. Each ord_v is non-Archimedean and non-trivial. Moreover, ord_v and ord_w induce different topologies on K if $v \neq w$. Since there are infinitely many codimension 1 $\overline{k}$-points of V, we thus have infinitely many non-trivial, non-Archimedean places of K. By construction, $\operatorname{ord}_v(f) = 0$ if and only if f and $1/f$ do not vanish along v. A non-zero function vanishes or has poles at only finitely many v, so $\operatorname{ord}_v(f) = 0$ for all but finitely many v. Moreover, $\operatorname{ord}_v(f) = 0$ for all v if and only if f is a non-zero constant.

The separability criterion for K_v/K follows from the fact that finite type schemes over a field are excellent [27, Scholie 7.8.3(iii) and Proposition 7.8.6(i)]. For a hands-on approach, we can modify Proposition 1.6 in the case of $K = \mathbf{F}_{p^m}(t_1, \dots, t_n)$. The completion K_v is the fraction field of the completion $\widehat{\mathscr{O}}_{V,v}$, so K_v is an intermediate field in the extension $\mathbf{F}_{p^m}((u_1, \dots, u_n))/K$ for some local coordinates $u_1, \dots, u_n$. As in Proposition 1.6, it suffices to show that $\mathbf{F}_s((u_1, \dots, u_n))/\mathbf{F}_s(u_1, \dots, u_n)$ is a separable extension with s a power of a prime p. Again by [24, Lemma 2.6.1 (b)], it suffices to prove the following proposition.

Proposition 1.8. *If $f_1, \dots, f_r \in \mathbf{F}_s((u_1, \dots, u_n))$ are linearly independent over $\mathbf{F}_s(u_1, \dots, u_n)$, then $f_1^p, \dots, f_r^p$ are linearly independent over $\mathbf{F}_s(u_1, \dots, u_n)$.*

Proof Suppose $\sum_{i=1}^r g_i f_i^p = 0$ for some $g_1, \dots, g_r \in \mathbf{F}_s(u_1, \dots, u_n)$. By clearing denominators, we may assume $f_1^p, \dots, f_r^p \in \mathbf{F}_s[[u_1, \dots, u_n]]$ and $g_1, \dots, g_r \in \mathbf{F}_s[u_1, \dots, u_n]$. Given $\mathbf{d} = (d_1, \dots, d_n) \in \mathbf{Z}^n_{\geq 0}$, let $\mathbf{u}^{\mathbf{d}} := u_1^{d_1} \cdots u_n^{d_n}$. Let $P = \{\mathbf{d} \in \mathbf{Z}^n_{\geq 0} : d_i < p \text{ for all } i\}$. Then there exist $\{g_{1,\mathbf{d}}, \dots, g_{r,\mathbf{d}}\}_{\mathbf{d} \in P} \subset \mathbf{F}_s[u_1, \dots, u_n]$ such that

$$g_i = \sum_{\mathbf{d} \in P} g_{i,\mathbf{d}}(u_1^p, \dots, u_n^p)\mathbf{u}^{\mathbf{d}}.$$

We write $g_{i,\mathbf{d}}(u_1^p, \dots, u_n^p) = h_{i,\mathbf{d}}^p$ for some $h_{i,\mathbf{d}} \in \mathbf{F}_s[u_1, \dots, u_n]$, so that

$$0 = \sum_{i=1}^{r} g_i f_i^p = \sum_{\mathbf{d} \in P} \left(\sum_{i=1}^{r} h_{i,\mathbf{d}}^p f_i^p \right) \mathbf{u}^{\mathbf{d}}.$$

Note that $\sum_{i=1}^{r} h_{i,\mathbf{d}}^p f_i^p \in \mathbf{F}_s[u_1^p, \dots, u_n^p]$ and $d_1, \dots, d_n < p$ for all $\mathbf{d}$. Following the proof of Proposition 1.6, we can thus conclude that $\left(\sum_{i=1}^{r} h_{i,\mathbf{d}} f_i\right)^p = 0$ for all $\mathbf{d} \in P$ by considering terms of multidegree $\mathbf{d}$ mod p. As in Proposition 1.6, it follows that $f_1^p, \dots, f_r^p$ are linearly independent over $\mathbf{F}_s(u_1, \dots, u_n)$. □

It remains to address the generalized product formula for K. For each $v \in V$ of codimension 1, we will construct a constant $0 < c_v < 1$ such that the absolute values $\| \cdot \|_v := c_v^{\mathrm{ord}_v(\cdot)}$ satisfy the generalized product formula. Since V is a projective k-variety, there is a closed embedding $i : V \hookrightarrow \mathbf{P}_k^n$ over k. Let $\overline{i(v)}$ be the closure of $i(v) \in \mathbf{P}_k^n$, so that $\overline{i(v)}$ is an integral closed subscheme of $\mathbf{P}_k^n$. Let $\deg_{k,i}(v)$ be the degree of $\overline{i(v)}$, and set

$$c_v := \begin{cases} |k|^{-\deg_{k,i}(v)} & |k| < \infty, \\ e^{-\deg_{k,i}(v)} & \text{otherwise.} \end{cases} \tag{2}$$

This choice of c_v allows us to deduce the generalized product formula geometrically. In particular, given a rational function $f \in K^\times$, the principal Weil divisor $\mathrm{div}(f) = \sum_{v \in V} \mathrm{ord}_v(f) \cdot v$ has degree 0. That is,

$$0 = \deg_{k,i}(\mathrm{div}(f)) = \sum_{v \in V} \mathrm{ord}_v(f) \deg_{k,i}(v),$$

so $\prod_{v \in V} \|f\|_v = c^0 = 1$, where $c = |k|$ if $|k| < \infty$ and $c = e$ otherwise.

Remark 1.9. If we are given a very ample line bundle L on V instead of a specified projective embedding $i : V \hookrightarrow \mathbf{P}_k^n$, we can still define a generalized global field structure on K. We simply replace $\deg_{k,i}(v)$ in Equation 2 with $\deg_{k,L}(v) := \deg_{k,\overline{v}}(c_1(L|_{\overline{v}})^{\dim \overline{v}})$.

2.2 Extensions of generalized global fields

We now discuss a generalized global field structure on finite extensions of generalized global fields. Let F be a finite extension of a generalized global field K. Since F/K is finite, each place v on K lifts to finitely many places w on F. Since at most finitely many places of K are Archimedean, these lift to the finitely many Archimedean places of F. Since each non-Archimedean place v is discretely valued, the same holds for each non-Archimedean lift w.

If $w(x) \neq 0$ for some $x \in F^\times$, then x (or $1/x$) is non-integral at w. This implies that one of the coefficients of the minimal polynomial of x (or $1/x$) over K is non-integral in the valuation ring of v. Since for each $y \in K^\times$ we have $v(y) = 0$ for all but finitely many v, it follows that $w(x) = 0$ for all but finitely many w.

By assumption, K_v/K is a separable extension for all non-Archimedean v. Given a non-Archimedean lift w of v, we need to check that F_w/F is a separable extension. Since F/K is finite, there are generators $\alpha_1, \ldots, \alpha_n \in F$ such that $F = K(\alpha_1, \ldots, \alpha_n)$. We will show that $F_w = K_v(\alpha_1, \ldots, \alpha_n)$. The separability of F_w/F will then follow from the separability of K_v/K.

Proposition 1.10. *Let v be a valuation on a field K. Let w be a valuation on the field $F = K(\alpha_1, \ldots, \alpha_n)$ that is an extension of v. Then $(K(\alpha_1, \ldots, \alpha_n))_w = K_v(\alpha_1, \ldots, \alpha_n)$.*

Proof First, we note that $\alpha_1, \ldots, \alpha_n \in F \hookrightarrow F_w$. We also have that F_w is an extension of K_v, so it follows that $K_v(\alpha_1, \ldots, \alpha_n) \subseteq F_w$. Since $K \subseteq K_v$, we have $F \subseteq K_v(\alpha_1, \ldots, \alpha_n) \subseteq F_w$, so $(K_v(\alpha_1, \ldots, \alpha_n))_w = F_w$. Finally, $K_v(\alpha_1, \ldots, \alpha_n)$ is complete with respect to w, since any finite extension of a complete valued field is complete with respect to the corresponding extension of the valuation. $\square$

We now choose a unique representative of each $\|\cdot\|_w$ by specifying

$$\|\cdot\|_w|_K = \|\cdot\|_v^{[F_w:K_v]e_v/e_w}. \tag{3}$$

For example, if v is Archimedean, then we are requiring $\|\cdot\|_w|_K = \|\cdot\|_v$. Indeed, if v is complex, then $F_w \cong K_v \cong \mathbf{C}$ and $e_v = e_w = 2$. If v and w are both real, then $F_w \cong K_v \cong \mathbf{R}$ and $e_v = e_w = 1$. If v is real and w is complex, then $[F_w : K_v] = [\mathbf{C} : \mathbf{R}] = 2$, while $e_v = 1$ and $e_w = 2$.

We need to check that our choices of $\|\cdot\|_w$ satisfy the generalized product formula. The trick here is to reduce to the generalized product formula over K using field norms. Since K_v/K is separable for all v, the ring $K_v \otimes_K F$ is reduced for all v. This induces an isomorphism

$$K_v \otimes_K F \to \prod_{w|v} F_w \quad \text{given by} \quad a \otimes b \mapsto (ab, \ldots, ab)$$

for all v. Also note that any basis of F as a K-vector space is also a basis of $K_v \otimes_K F$ as a K_v-vector space. Given $x \in F^\times$, we thus have

$$N_{F/K}(x) = N_{(K_v \otimes_K F)/K_v}(x) = \prod_{w|v} N_{F_w/K_v}(x),$$

where $N_{E'/E}$ is the norm of the extension E'/E. In particular,

$$\prod_{w|v} \|N_{F_w/K_v}(x)\|_w = \|N_{F/K}(x)\|_v$$

for all v. Since $\|x\|_w = \|N_{F_w/K_v}(x)\|_w^{1/[F_w:K_v]}$, it follows that

$$\begin{aligned}\prod_{w|v} \|x\|_w^{e_w} &= \prod_{w|v} \|N_{F_w/K_v}(x)\|_w^{e_w/[F_w:K_v]} \\ &= \prod_{w|v} \left(\|N_{F_w/K_v}(x)\|_v^{e_w/[F_w:K_v]}\right)^{[F_w:K_v]e_v/e_w} \\ &= \prod_{w|v} \|N_{F_w/K_v}(x)\|_v^{e_v} = \|N_{F/K}(x)\|_v^{e_v}.\end{aligned}$$

The generalized product formula for $\{\|\cdot\|_w\}_w$ on F thus follows from the generalized product formula for $\{\|N_{F/K}(\cdot)\|_v\}_v$ on K.

3 Heights

Given an algebraic variety X over a field k, a *height* is a function $h : X(\overline{k}) \longrightarrow \mathbf{R}_{\geq 0}$, with $h(x)$ a measure of the complexity of x. Using the ordering on $\mathbf{R}$, we can filter $X(\overline{k})$ by height, which allows us to study rational points using limits and induction. Ideally, one would like points of bounded height and bounded degree to be finite sets. This property (known as the *Northcott property*) holds for many, but not all, of the height functions that we will describe.

Following [18, Part B] and [12, Section 9], we discuss a few classical height functions. A central theme is that valuations and the product formula are useful in constructing heights, both for global and generalized global fields. To conclude this section, we briefly describe a geometric approach to heights over finitely generated fields of transcendence degree 1 over $\mathbf{Q}$. These geometric heights will serve as an analogy for Moriwaki's Arakelov-theoretic heights discussed in Section 6.

3.1 Naïve and logarithmic heights

Any non-zero element of $\mathbf{Q}$ can be written uniquely as a fraction $\frac{a}{b}$, where $a, b \in \mathbf{Z}$, $b > 0$, and $\gcd(a, b) = 1$. We define the *naïve height* of $\frac{a}{b}$ to be $h(\frac{a}{b}) = \max\{|a|, |b|\}$. For scaling reasons and to ensure that the minimum value attained by the height is 0, one defines the *logarithmic* *height* $\log h(\frac{a}{b}) = \log\max\{|a|, |b|\}$. We can mimic these definitions to obtain our first height functions on $\mathbf{P}^n(\mathbf{Q})$.

Definition 1.11. Any rational point $x \in \mathbf{P}^n(\mathbf{Q})$ can be written uniquely (up to scaling coordinates by ± 1) as $x = [x_0 : \ldots : x_n]$, where $x_0, \ldots, x_n \in \mathbf{Z}$ and $\gcd(x_0, \ldots, x_n) = 1$. The *naïve (multiplicative) height* and *naïve logarithmic height* of x are defined to be $h(x) := \max\{|x_0|, \ldots, |x_n|\}$ and $\log h(x)$, respectively. Note that

$$\max\{|x_0|, \ldots, |x_n|\} = \max\{|-x_0|, \ldots, |-x_n|\},$$

so these heights are well-defined.

Remark 1.12. Since $\{n \in \mathbf{Z} : |n| \leq H\}$ is a finite set for any positive bound H, it follows that sets of bounded naïve or logarithmic height are finite. This is known as the *Northcott property*. Later, we will discuss how to define height functions on varieties over number fields. A height function h on a variety V over a number field K is said to satisfy the *Northcott property* if for any real numbers $D, H > 0$, the set

$$\{x \in V(\overline{\mathbf{Q}}) : [\mathbf{Q}(x) : \mathbf{Q}] < D \text{ and } h(x) < H\}$$

is finite. This is a desirable property for many applications, and plays a central role in results such as the Mordell–Weil or Lang–Néron theorems.

We now define naïve and logarithmic heights on $\mathbf{P}^n(K)$ for any number field K using the global field structure on K, as in Proposition 1.5. Let M_K denote the set of places on K.

Definition 1.13. Let K be a number field, and let $x = [x_0 : \ldots : x_n] \in \mathbf{P}^n(K)$. The *naïve (multiplicative) height* and *naïve logarithmic height* of x with respect to K are defined to be

$$h_K(x) := \prod_{v \in M_K} \max\{\|x_0\|_v, \ldots, \|x_n\|_v\}$$

and $\log h_K(x)$, respectively.

Remark 1.14. The naïve multiplicative and logarithmic heights with respect to K are well-defined by the product formula. Indeed, for any $c \neq 0$,

$$\begin{aligned} \prod_{v \in M_K} \max_{0 \leq i \leq n}\{\|cx_i\|_v\} &= \left(\prod_{v \in M_K} \|c\|_v\right)\left(\prod_{v \in M_K} \max_{0 \leq i \leq n}\{\|x_i\|_v\}\right) \\ &= \prod_{v \in M_K} \max_{0 \leq i \leq n}\{\|x_i\|_v\}. \end{aligned}$$

Going beyond just number fields, we would like to define a notion of height on $\mathbf{P}^n(\overline{\mathbf{Q}})$. To do this, we will have to keep track of the field of definition of a given $\overline{\mathbf{Q}}$-rational point of $\mathbf{P}^n$. Given a finite extension F/K, we can naturally view $\mathbf{P}^n(K)$ as a subset of $\mathbf{P}^n(F)$. For any $x \in \mathbf{P}^n(K)$, one can show that $h_F(x) = h_K(x)^{[F:K]}$ [18, Lemma B.2.1 (c)].

Definition 1.15. Let $x \in \mathbf{P}^n(\overline{\mathbf{Q}})$. The *absolute (multiplicative) height* and *absolute logarithmic height* of x are defined to be

$$h_{\mathrm{abs}}(x) = h_K(x)^{1/[K:\mathbf{Q}]},$$
$$\log h_{\mathrm{abs}}(x) = \frac{1}{[K:\mathbf{Q}]} \log h_K(x),$$

respectively, where K is any number field over which x is defined.

As one would hope, the absolute height satisfies a Northcott property, albeit in a slightly different form than for the naïve height on $\mathbf{P}^n(\mathbf{Q})$. We will see that we need to bound both the height and degree of the field of definition to get a finite set of points.

Theorem 1.16. *[18, Theorem B.2.3] For any $H, D \geq 0$, the set*

$$\{x \in \mathbf{P}^n(\overline{\mathbf{Q}}) : h_{\mathrm{abs}}(x) \leq H \text{ and } [\mathbf{Q}(x) : \mathbf{Q}] \leq D\}$$

is finite.

It follows that for any fixed number field K, h_K and $\log h_K$ satisfy the Northcott property. The absolute height is also invariant under Galois action [18, Proposition B.2.2].

3.2 Weil heights

Given a projective variety X over a number field K with a very ample line bundle L, we get an embedding $\phi : X \to \mathbf{P}^n_K$. This enables us to define a height $\log h_{L,K} : X(K) \to \mathbf{R}_{\geq 0}$ by setting $\log h_{L,K}(x) := \log h_K(\phi(x))$ (and similarly for $\log h_{L,\mathrm{abs}}$). Of course, one needs to ask how this depends on the embedding ϕ; it turns out that $\log h_{L,K}$ is well-defined *up to a bounded function* [18, Theorem B.3.1]. This leads us to the notion of *Weil heights*.

Notation 1.17. Given any set S, let $O(1)$ be the set of bounded functions $S \to \mathbf{R}$. Given a function $f : S \to \mathbf{R}$, we denote set of functions $\{g : S \to \mathbf{R} : g - f \text{ bounded on } S\}$ by $f + O(1)$.

Definition 1.18. Let X be a projective variety over a number field K with a line bundle L. Then there exist very ample line bundles L_1, L_2 such that $L \cong L_1 \otimes L_2^{-1}$. The *Weil height* with respect to L is the difference

$$\log h_{K,L} := \log h_{K,L_1} - \log h_{K,L_2} + O(1) : X(K) \to \mathbf{R}.$$

Similarly, we define the *absolute Weil height* to be

$$\log h_{\mathrm{abs},L} := \log h_{\mathrm{abs},L_1} - \log h_{\mathrm{abs},L_2} + O(1) : X(\overline{\mathbf{Q}}) \to \mathbf{R}.$$

The differences $\log h_{K,L_1} - \log h_{K,L_2}$ and $\log_{h_{\mathrm{abs},L_1}} - \log h_{\mathrm{abs},L_2}$ depend on the choice of L_1, L_2 only up to $O(1)$, so the Weil height and absolute Weil height are well-defined.

As with absolute naïve heights, the absolute Weil height is invariant under Galois actions.

Remark 1.19. Weil heights are additive in L. That is, given two line bundles L, L', we have $\log h_{K,L\otimes L'} = \log h_{K,L} + \log h_{K,L'}$ (and likewise for absolute Weil heights).

Remark 1.20. In some circumstances, there is a particular representative of a Weil height in its $O(1)$-equivalence class that satisfies nice properties. The *canonical* or *Néron–Tate height* is an important example of such a height function [18, Section B.4].

3.3 Heights over generalized global fields

When constructing heights on projective varieties over a number field K, we saw that the global field structure of K played an essential role. The defining characteristics of generalized global fields encapsulate the properties of a global field that allow one to construct height functions. Following [12, Section 9], we can construct height functions on projective varieties over generalized global fields in a manner analogous to the height functions discussed thus far. We start with a generalization of absolute (logarithmic) heights on $\mathbf{P}^n(\overline{\mathbf{Q}})$.

Definition 1.21. Let $(K, \{\|\cdot\|_v\}_v)$ be a generalized global field. The *standard K-height* and *logarithmic K-height* are functions $\mathbf{P}^n(\overline{K}) \to \mathbf{R}_{\geq 0}$ defined by

$$H_K(x) = \prod_w \max\{\|x_0\|_w^{e_w/[F:K]}, \ldots, \|x_n\|_w^{e_w/[F:K]}\},$$

$$\log H_K(x) = \frac{1}{[F:K]} \sum_w \log \max\{\|x_0\|_w^{e_w}, \ldots, \|x_n\|_w^{e_w}\},$$

respectively, where F is any finite extension of K over which x is defined, endowed with a generalized global field structure as described in Section 2.2.

Remark 1.22. Let F' be a finite extension of F. Since $[F' : K] = \sum_{w'|w}[F'_{w'} : F_w]$ for all w on F, Equation 3 implies that H_K and $\log H_K$ do not depend on the choice of field of definition F. Moreover, the generalized product formula implies that $H_K(x)$ and $\log H_K(x)$ do not depend on the choice of projective coordinates of x (compare to Remark 1.14). One can also prove $\mathrm{Aut}(\overline{K}/K)$-invariance, so that the K-height does not depend on the choice of algebraic closure $\overline{K}$.

Remark 1.23. Note that $\log H_{\mathbf{Q}} = h_{abs}$ on $\mathbf{P}^n(\overline{K})$.

We now extend the definition of absolute Weil heights to generalized global fields. Given a projective variety X over a field K with a very ample line bundle L, let $H_{K,L} = H_K \circ \phi$, where $\phi : X \to \mathbf{P}^n_K$ is any projective embedding determined by L.

Definition 1.24. Let K be a generalized global field. Let X be a projective variety over K with a line bundle L. Let L_1, L_2 be very ample line bundles on X such that $L \cong L_1 \otimes L_2^{-1}$. The *generalized Weil height* is defined as

$$\log H_{K,L} := \log H_{K,L_1} - \log H_{K,L_2} + O(1) : X(\overline{K}) \to \mathbf{R}.$$

Generalized Weil heights satisfy many nice properties. For example, generalized Weil heights are additive in L (see Remark 1.19). Moreover, generalized Weil heights are functorial: given a map $f : X \to Y$ of projective K varieties and a line bundle L on Y, we have

$$\log H_{K,f^*L} = \log H_{K,L} \circ f + O(1)$$

as functions on $X(\overline{K})$.

3.4 Geometric heights

We now discuss a method for defining heights in terms of the degree of a line bundle. This approach will be mirrored by Moriwaki's height function, which we describe in Section 6. Let K be a finitely generated field of transcendence degree 1 over a prime field k. Let C be a curve over k such that $k(C) = K$. A point $x \in \mathbf{P}^n(K)$ determines a map $\phi_x : C \to \mathbf{P}^n$. The pullback $\phi_x^* \mathcal{O}_{\mathbf{P}^n}(1)$ is a line bundle on C.

Definition 1.25. The *geometric height* of $x \in \mathbf{P}^n(K)$ is $h_{\mathrm{geom}}(x) := \deg(\phi_x^* \mathcal{O}_{\mathbf{P}^n}(1))$.

Remark 1.26. Let M'_K be the places on K that are trivial on k (which correspond to the codimension 1 points of C). Given coordinates $x = [x_0 : \ldots : x_n]$, we have $h_{\mathrm{geom}}(x) = -\sum_{v \in M'_K} \min\{\mathrm{ord}_v(x_0), \ldots, \mathrm{ord}_v(x_n)\}$.

To define the geometric height of points in $\mathbf{P}^n(\overline{K})$, we must keep track of the field of definition as we did for absolute heights. Any point $x \in \mathbf{P}^n(\overline{K})$ is defined over $k(C')$ for some finite cover $C' \to C$ of degree $[k(C') : k(C)]$. This defines a map $\phi_x : C' \to \mathbf{P}^n$, and we again get a line bundle $\phi_x^* \mathscr{O}_{\mathbf{P}^n}(1)$ on C'.

Definition 1.27. The *geometric height* of $x \in \mathbf{P}^n(\overline{K})$ is

$$h_{\mathrm{geom}}(x) := \frac{\deg(\phi_x^* \mathscr{O}_{\mathbf{P}^n}(1))}{[k(C') : k(C)]},$$

where C' is any finite cover of C such that x is defined over $k(C')$.

Finally, let X be a projective variety over K. Let L be an ample line bundle on X. A point $x \in X(\overline{K})$ determines a map $\phi_x : C' \to X$, where $C' \to C$ is a finite cover of degree $[k(C') : k(C)]$. As before, $\phi_x^* L$ is a line bundle on C'.

Definition 1.28. The *geometric height* of $x \in X(\overline{K})$ is

$$h_{\mathrm{geom},L}(x) := \frac{\deg(\phi_x^* L)}{[k(C') : k(C)]},$$

where C' is any finite cover of C such that x is defined over $k(C')$.

Remark 1.29. The geometric height $h_{\mathrm{geom},L} : X(\overline{K}) \to \mathbf{R}$ depends on the choice of ample line bundle L.

Remark 1.30. The assumption that L is ample is a *positivity assumption*. Because L is ample and ϕ_x is finite, $\phi_x^* L$ is an ample bundle on the curve C'. By the Riemann–Roch theorem, a line bundle on a curve is ample if and only if its degree is positive, so the assumption that L is ample guarantees that $h_{\mathrm{geom},L} : X(\overline{K}) \to \mathbf{R}$ only takes non-negative values. We will see analogous positivity assumptions on the line bundles used to define Moriwaki heights in Section 6.

4 Analytic background

In order to construct geometric heights over more general fields of characteristic 0, one must make sense of the degree of a line bundle at the infinite place. In particular, one needs an intersection theory on the finite and infinite fibers of maps of the form $\mathcal{X} \to \operatorname{Spec} \mathbf{Z}$. Arakelov laid the groundwork for understanding intersection theory on surfaces in an arithmetic sense (even at the infinite place, which a priori only had a complex structure) [17]. This theory was developed further by Faltings in [28], and later generalized to higher dimensional varieties by Gillet and Soulé (see e.g. [19]). We will provide some analytic background for arithmetic intersection theory, following [19] and [29]. In this section, we restrict our attention to complex manifolds. All complex manifolds arising in our context will be algebraic varieties.

4.1 Differential forms

Let X be a complex manifold of dimension n. Let U be an open subset of X isomorphic to $\mathbf{C}^n$. Pick a system of local coordinates $z_1, z_2, \ldots, z_n$ on U and write $z_j = x_j + iy_j$. A function $f : U \to \mathbf{C}$ is said to be *homolorphic* if it satisfies the Cauchy–Riemann equations with respect to each pair (x_j, y_j). A function is holomorphic on X if it is holomorphic on each chart. Holomorphic functions are infinitely ($\mathbf{R}$-)differentiable, and we denote by $C^\infty(X)$ the class of infinitely $\mathbf{R}$-differentiable functions on X. The structure sheaf $\mathcal{O}_X$ of X, is the sheaf of holomorphic functions on X. A holomorphic vector bundle on X is a vector bundle $p : E \to X$ such that (i) p is holomorphic, and (ii) the local trivializations $p^{-1}(U) \cong U \times \mathbf{C}^{\operatorname{rank}(E)}$ are biholomorphic maps.

Definition 1.31 (Complexified tangent bundle)**.** Let X be a complex manifold and let TX denote the tangent bundle on the underlying real manifold. The complexified tangent bundle is $T_{\mathbf{C}}X := TX \otimes \mathbf{C}$.

The complexified tangent bundle admits a decomposition

$$T_{\mathbf{C}}X = T^{1,0}X \oplus T^{0,1}X$$

of complex vector bundles on X. The bundle $T^{1,0}X$ is naturally isomorphic to the holomorphic tangent bundle of X, while the antiholomorphic tangent bundle $T^{0,1}X$ of X is complex conjugate to $T^{1,0}X$.

Remark 1.32. In contrast to the analytic nature of Definition 1.31, one can define the holomorphic tangent bundle in a more algebraic manner. Define the holomorphic cotangent bundle Ω^1_X by setting $\Omega^1_X(U)$ to be the $\mathcal{O}_X(U)$-algebra generated by

$$\{df : f \in \mathcal{O}_X(U) \text{ and } df \text{ satisfies the Leibniz rule}\}.$$

Then $T^{1,0}X$ is the dual of Ω^1_X.

For any integer k, let $A^k(X) := \bigwedge^k (T_{\mathbf{C}}X)^*$, where $(-)^*$ denotes the dual. This sheaf is often called the *space of k-forms* on X.

Definition 1.33 (Differential (p,q)-forms)**.** Let $p, q \in \mathbf{Z}_{\geq 0}$. Define the *sheaf of (p,q)-forms* as $A^{p,q}(X) := \left(\bigwedge^p (T^{1,0}X)^*\right) \otimes \left(\bigwedge^q (T^{0,1}X)^*\right)$.

The sheaf $A^{p,q}(X)$ has an explicit description in local coordinates. Let $U \subset X$ be an open set with local coordinates $z_1, z_2, \ldots, z_n$. A differential (p,q)-form on U is a $\mathbf{C}(U)$-linear combination of the form

$$\sum a_{i_1 i_2 \ldots j_q} dz_{i_1} \wedge dz_{i_2} \ldots \wedge dz_{i_p} \wedge d\overline{z}_{j_1} \wedge d\overline{z}_{j_2} \ldots \wedge d\overline{z}_{j_q},$$

where $\overline{z}$ denotes complex conjugation, and the sum is over all tuples of size p and q.

In subsequent sections, we will use the maps $\partial : A^{p,q}(X) \to A^{p+1,q}(X)$ and $\overline{\partial} : A^{p,q}(X) \to A^{p,q+1}(X)$, which are given on coordinate charts by $\partial(f\,\omega) = \sum_{k=1}^n \frac{\partial f}{\partial z_k}\, dz_k \wedge \omega$ and $\overline{\partial}(f\,\omega) = \sum_{k=1}^n \frac{\partial f}{\partial \overline{z}_k}\, d\overline{z}_k \wedge \omega$.

Remark 1.34. The maps ∂ and $\overline{\partial}$ are closely related to the exterior derivative. For a local function f, the exterior derivative is defined as $d(f) = \sum(\partial f/\partial z_i) dz_i + \sum(\partial f/\partial \overline{z}_j) dz_j$. This can be extended to a map $d : A^k(X) \to A^{k+1}(X)$ using the Leibniz rule: $d(u \wedge v) = du \wedge v + (-1)^{\deg u} u \wedge dv$. Using the decomposition $A^k(X) = \bigoplus_{p+q=k} A^{p,q}(X)$, we have that $d = \partial + \overline{\partial}$. For more details, see [29].

Hermitian metrics

Before we proceed, we recall a few definitions about Hermitian vector bundles on a manifold.

Definition 1.35. A *Hermitian form* on a complex vector space V is a pairing $H : V \times V \to \mathbf{C}$ such that (i) $H(u,v)$ is $\mathbf{C}$-linear in the first variable, and (ii) $H(u,v) = \overline{H(v,u)}$ for all $u, v \in V$. Further, H is *positive definite* if $H(u,u) > 0$ for all $u \neq 0$. In this case, one can associate a metric to a Hermitian form by defining $\|u\|_H := \sqrt{H(u,u)}$. In what follows, we will suppress the subscript H whenever it is clear from context.

Definition 1.36. A *Hermitian metric H* on a holomorphic vector bundle $E \to X$ on a complex manifold X is a smoothly varying positive definite Hermitian form on each fiber. A *(Hermitian) metrized vector bundle* on X is a pair (E, H) of a vector bundle E equipped with a (Hermitian) metric H.

Example 1.37. Let $(X, L) = (\mathbf{P}^n(\mathbf{C}), \mathcal{O}(1))$. For any point $x = [x_0 : \ldots : x_n]$ and section $s \in L$ that doesn't vanish in a neighborhood of x, define

$$\|s(x)\|_\infty = \frac{|s(x)|}{\max\{|x_0|, |x_1| \ldots |x_n|\}}.$$

Then $(L, \|\cdot\|_\infty)$ is a metrized line bundle on X.

Every complex vector bundle admits a Hermitian metric by gluing together the standard Hermitian metric on $\mathbf{C}^n$ (see [29, Proposition 4.1.4]).

4.2 Currents

A *current* is an element of the dual space of the space of differential forms, that satisfies some additional completeness properties. In this article, we will not define currents in full generality. Instead, we will give some key examples that are sufficient for the purpose of this article. We denote by $D_{p,q}(X) := (A^{p,q}(X))^*$ and $D_d(X) := (A^d(X))^*$ the space of currents of bidimension (p, q) and the space of currents of dimension d, respectively.

Example 1.38 (Current associated to a subspace)**.** Let $\iota : Y \to X$ be an analytic subspace of X of dimension k, and let $\alpha \in A^{2k}(X)$ be a differential form on X. We define a current $\delta_Y \in D_{2k}(X)$ by

$$\delta_Y(\alpha) = \int_Y \iota^* \alpha.$$

Note that this definition can be extended to any analytic cycle, i.e. any $\mathbf{Z}$-linear combination of analytic subspaces. Also note that if $\beta \in A^{p,q}(X)$ with $p + q = 2k$, then $\iota^*(\beta) = 0$ unless $p = q = k$. It follows that $\delta_Y \in D_{k,k}(X)$.

Example 1.39 (Current associated to a differential form)**.** Let α denote a (p, q)-form. The current associated to α is the map

$$[\alpha] : A^{n-p,n-q}(X) \to \mathbf{C} \quad \text{given by} \quad \beta \mapsto \int_X \alpha \wedge \beta.$$

This defines a map $A^{p,q}(X) \to D_{n-p,n-q}(X)$ sending α to $[\alpha]$. Alternatively, one may think of this as a pairing:

$$A^{p,q}(X) \times A^{n-p,n-q}(X) \to \mathbf{C} \quad \text{given by} \quad (\alpha, \beta) \mapsto \int_X \alpha \wedge \beta.$$

Example 1.40 (Logarithmic current associated to a line bundle)**.** Let Y be a divisor on X, and let L be the line bundle corresponding to Y. Let s be a section of L. Choose a smooth Hermitian norm $\|\cdot\|$ on L. Then $\log\|s\|^2$ is a $(0,0)$-form on X, which has an associated current $[-\log\|s\|^2]$. Further, $[-\log\|s\|^2]$ is a *Green current* for Y; that is, there exists a smooth closed (1,1)-form β on X such that

$$\frac{i}{2\pi}\partial\overline{\partial}\log\|s\|^2 = \delta_Y - \beta.$$

The form $\beta \in A^{1,1}(X)$ is known as the *Chern form* and will be discussed in the following section.

4.3 Chern classes

Given a line bundle L on X, one can define its first Chern class $c_1(L)$ in a variety of ways. In arithmetic intersection theory, one needs both the algebraic and the analytic description of the Chern class. In this section, we give a brief analytic description of $c_1(L)$. Let us first recall the definition of $c_1(L)$.

Definition 1.41 (First Chern class)**.** Let $L \in H^1(X, \mathscr{O}_X^*)$ be a line bundle. Then $c_1(L)$ is the image of L in $H^2(X, \mathbf{Z})$ under the boundary map of the long exact sequence induced by the exponential exact sequence $0 \to \mathbf{Z} \to \mathscr{O}_X \to \mathscr{O}_X^* \to 0$.

Definition 1.42. Let X be a manifold with a metrized Hermitian holomorphic line bundle L on X, and let s be a section of L. The *first Chern class*, which by abuse of notation we also denote by $c_1(L)$, is the de Rham cohomology class of the differential form whose associated current is given by

$$\delta_{\mathrm{div}(s)} - \frac{i}{2\pi}\partial\overline{\partial}[\log\|s\|^2]. \tag{4}$$

This is independent of the choice of s by the Poincaré–Lelong formula ([30, Chapter 3, Section 2]), which states that $\delta_{\mathrm{div}(f)} + \frac{i}{2\pi}\partial\overline{\partial}[\log\|f\|^2] = 0$ for any meromorphic function f on X. Since $\log\|s\| \in A^{0,0}(X)$ and $\delta_{\mathrm{div}(s)} \in D_{n-1,n-1}(X)$, we have $c_1(L) \in A^{1,1}(X)$.

Remark 1.43. We give a brief justification for the abuse of notation in Definition 1.42.

(i) Recall that the de Rham cohomology of X is defined as the cohomology of the complex $A^\bullet(X)$. Further, $c_1(L)$ is closed and invariant under complex conjugation and thus defines a cohomology class in $H^2_{dR}(X, \mathbf{R}) \subset H^2_{dR}(X, \mathbf{C})$. The divisor $\mathrm{div}(s)$ also defines a class in $H^2(X, \mathbf{R})$ via the map $H^2(X, \mathbf{Z}) \to H^2_{dR}(X, \mathbf{R})$. The Poincaré–Lelong formula can be used to show that these two classes in $H^2_{dR}(X, \mathbf{R})$ are the same (see e.g. [29, Proposition 4.4.123]).

(ii) The "analytic" Chern class is usually not defined as in Equation 4, but rather as the failure of a certain complex to be exact. This approach gives an explicit way to calculate $c_1(L)$. We omit the details here for brevity and refer the reader to [29, Chapter 4] for details.

4.4 Arakelov–Green currents

In this section, we briefly define Arakelov–Green functions on a Riemann surface X. These functions were used by Arakelov in [17] to define an Archimedean version of the local intersection number. In essence, they play the same role as a uniformizer in the non-Archimedean case. This comes up in Section 5. See [31] for more details.

Definition 1.44. Let X be a Riemann surface and let μ be a Hermitian metric on X with volume element $d\mu$. The *Arakelov–Green function* $G : X \times X \to \mathbf{R}_{\geq 0}$ for μ is the unique function satisfying all of the following properties:

(i) $G(P, Q)^2 \in C^\infty(X \times X)$ and vanishes only on the diagonal Δ_X. For a fixed $P \in X$, an open neighborhood U of P, and a local coordinate z on U, there exists $f \in C^\infty(X)$ such that $\log G(P, Q) = \log(z(Q)) + f(Q)$ for all $Q \in U \backslash \{P\}$.

(ii) For all $P \in X$, we have $\partial_Q \overline{\partial}_Q \log(G(P, Q))dxdy = 2\pi\imath d\mu(Q)$ for any $Q \neq P$.

(iii) G is symmetric, i.e. $G(P, Q) = G(Q, P)$.

(iv) For all $P \in X$, we have $\int_X \log G(P, Q) d\mu(Q) = 0$.

Condition (i) allows us to think of $G(P, -)$ as a uniformizer around P. The rest of the conditions uniquely determine G among the class of possible uniformizers.

Example 1.45. Let $X = \mathbf{P}^1_{\mathbf{C}}$ with the metric be given by $d\mu = \frac{1}{2\pi} \frac{|dz|^2}{(1+|z|^2)^2}$. (This is a normalized version of the Fubini–Study metric.) Then the corresponding Green function is given by

$$G^2(w, z) = e \frac{|w - z|^2}{(1 + |w|^2)(1 + |z|^2)}.$$

Remark 1.46. A Green function defines a Hermitian metric on the line bundle $\mathcal{O}_{X\times X}(\Delta_X)$ via $\|s_\Delta\| = G(P,Q)$, where s_Δ is the image of the unit section of $\mathcal{O}_{X\times X}$.

5 Arithmetic intersection theory and Arakelov theory

We now describe intersection theory on arithmetic varieties. Roughly speaking, arithmetic varieties are varieties over rings which have both finite and infinite places (e.g. the ring of integers over a number field). This differs from intersection theory in more classical settings (e.g. as in [21]) in that it takes into account the sizes of the residue fields at the finite places as well as makes sense of what it means for divisors to "intersect at infinity."

We begin by giving some definitions in §5.1. In §5.2, we describe intersection theory on surfaces following [17]. In the remaining subsections, we give a description of intersections on higher dimensional arithmetic varieties following [19] and [13].

5.1 Arithmetic varieties

The main reference for this section is [19]. Arithmetic varieties in Gillet and Soulé are defined over *arithmetic rings*, which are essentially rings equipped with embeddings into $\mathbf{C}$ and a notion of complex conjugation. For the purpose of this section we will consider arithmetic varieties over the ring of integers in number field K. We will let $B = \operatorname{Spec}\mathcal{O}_K$, where $\mathcal{O}_K$ denotes the ring of integers of K.

Definition 1.47. An *arithmetic variety* $\mathcal{X}$ over B is a flat, finite type scheme over B. We write $\mathcal{X}_K$ for the generic fiber of $\mathcal{X}$. For any point $s \in B$, we denote by $\mathcal{X}_s$ the fiber over s. If $\sigma : \mathcal{O}_K \to \mathbf{C}$ is an embedding of $\mathcal{O}_K$, we write $\mathcal{X}_\sigma := \mathcal{X} \otimes_{\sigma,\mathcal{O}_K} \mathbf{C}$. If Σ denotes the set of embeddings of $\mathcal{O}_K$ into $\mathbf{C}$, we write $\mathcal{X}_\Sigma := \coprod_{\sigma\in\Sigma} \mathcal{X}_\sigma$. The analytic subspace $\mathcal{X}_\Sigma(\mathbf{C})$ comes equipped with an involution, which we call F_∞. An arithmetic surface is an arithmetic variety $\mathcal{X} \to B$ such that the generic fiber $\mathcal{X}_K$ is a geometrically connected curve over K.

From now on, we will assume that $\mathcal{X}_K$ is smooth for convenience.

Example 1.48. $\mathbf{P}^n$ is an arithmetic variety over $\mathbf{Z}$, where the infinite fiber is the complex manifold $\mathbf{P}^n(\mathbf{C})$. In this case, F_∞ is just complex conjugation on the coordinates.

Example 1.49 (Néron model)**.** The Néron model of a rational elliptic curve is an arithmetic variety over $\mathbf{Z}$. More precisely, given an elliptic curve $E_{/\mathbf{Q}}$, there exists a smooth (commutative group) scheme $\mathcal{E}$ over $\mathbf{Z}$ whose generic fiber $\mathcal{E}_{\mathbf{Q}}$ is isomorphic to E. This scheme $\mathcal{E}_{/\mathbf{Z}}$ is called the *Néron model* of E. Its existence is a deep theorem. For more details, we refer the reader to [32] for the elliptic curve case and to [33] for the case of general abelian varieties. In the elliptic curve case, the infinite fiber is a torus, and F_∞ is the complex conjugation induced from $\mathbf{C}$.

We will write $A^{p,q}(\mathcal{X})$ for $\bigoplus_{\sigma\in\Sigma} A^{p,q}(\mathcal{X}_\sigma)$. Any integral subscheme Y of $\mathcal{X}$ of pure dimension is an arithmetic variety in its own right. In particular, $Y_\Sigma(\mathbf{C})$ is also a (disjoint union of) complex manifold(s).

Definition 1.50. An *Arakelov divisor* on $\mathcal{X}$ is the sum of a Weil divisor on $\mathcal{X}$ and an infinite contribution $\sum_\sigma \alpha_\sigma \mathcal{X}_\sigma$, where the sum is over all embeddings $\sigma : K \hookrightarrow \mathbf{C}$.

Let $\mathcal{X}$ be an arithmetic variety, with a choice of volume form μ_σ on each infinite fiber $\mathcal{X}_\sigma$. Let D be a Weil divisor on $\mathcal{X}$. Then, a choice of Green currents (see §4.2) $\{[g_\sigma]\}_{\sigma\in\Sigma}$ for D_σ on $\mathcal{X}_\Sigma$ turns D into an Arakelov divisor. In this case, $\alpha_\sigma = \int g_\sigma \cdot d\mu_\sigma$.

Definition 1.51. A *principal* Arakelov divisor is of the form

$$\operatorname{div}(f) + \sum_{\sigma\in\Sigma} \nu_\sigma(f)\mathcal{X}_\sigma$$

for a rational function f on $\mathcal{X}$, where $\nu_\sigma(f) := -\int_{\mathcal{X}_\sigma} \log|f|_\sigma \cdot d\mu_\sigma$.

Let $\pi : \mathcal{X} \to B$ be an arithmetic surface. For any point $s \in B$ (or infinite place σ) the fiber $\mathcal{X}_s$ (resp. $\mathcal{X}_\sigma$) is a *vertical* divisor. In general, a *vertical* divisor is a linear combination of such fibers. An irreducible *horizontal* divisor is a divisor D such that $\pi(D) = B$. In particular, if D is a horizontal divisor, then there is a finite extension F/K and a map $\varepsilon : \operatorname{Spec}\mathcal{O}_F \to \mathcal{X}$ over B such that $D = \varepsilon(\operatorname{Spec}\mathcal{O}_F)$.

5.2 Intersections on an arithmetic surface

We now discuss intersections on an arithmetic surface (i.e. a two dimensional scheme whose generic fiber is a smooth curve) as motivation for intersections on higher dimensional varieties. We primarily follow [17]. A discussion on intersections on arithmetic surfaces can also be found in [34, Section 9.1]. For a detailed exposition on Arakelov theory for arithmetic surfaces, we refer the reader to [31]. We will assume that $\pi : \mathcal{X} \to B = \operatorname{Spec} \mathcal{O}_K$ is an arithmetic surface that is regular in codimension 1 with smooth generic fiber $\mathcal{X}_K$.

Definition 1.52. Let D_1, D_2 be distinct irreducible divisors on $\mathcal{X}$ and let $x \in \mathcal{X}$ be a closed point. Let f and g be two functions that cut out D_1 and D_2 locally around x. We define

$$\langle D_1, D_2 \rangle_x := \operatorname{length}_{\mathcal{O}_{\mathcal{X},x}} (\mathcal{O}_{\mathcal{X},x}/(f,g)) \log |k(x)|,$$

where $k(x)$ denotes the residue field of x. The first part of this intersection number is "geometric" in nature, in that it looks like the intersection number in the algebraically closed case [21]. The second part, $\log |k(x)|$, keeps track of arithmetic information about the points of intersection. This intersection pairing can be extended by linearity to any pair of divisors on $\mathcal{X}$.

Any point $x \in \mathcal{X}$ is in $\mathcal{X}_b$ for some $b \in B$. Further, the residue field $k(x)$ of x, is a finite extension of the residue field $k(b)$ of b. Let D_1 and D_2 be two divisors on $\mathcal{X}$ with no common components. Define the *total intersection* over b as

$$\langle D_1, D_2 \rangle_b := \sum_{x \in |D_1 \cap D_2|} \langle D_1, D_2 \rangle_x.$$

Lemma 1.53 ([17], Section 1)**.** *Let D_1 be a horizontal divisor on $\mathcal{X}$, i.e. there is some finite extension F/K with ring of integers $\mathcal{O}_F$ such that D_1 is the image of a map $\varepsilon : \operatorname{Spec} \mathcal{O}_F \to \mathcal{X}$. Let D_2 be any other divisor on $\mathcal{X}$. Let $x \in D_1 \cap D_2$ be a closed point of $\mathcal{X}$ and suppose D_2 is defined locally around x by a function f. Let $\mathfrak{p}_1, \ldots \mathfrak{p}_r$ be the primes of $\mathcal{O}_F$ such that $\varepsilon(\mathfrak{p}_i) = x$. Then*

$$\langle D_1, D_2 \rangle_x = \sum_{i=1}^{r} -\log \|\varepsilon^* f\|_{\mathfrak{p}_i}.$$

Proof Let $f|_{D_1}$ denote the restriction of f to D_1. Then by definition of the intersection number, we have $\langle D_1, D_2 \rangle_x = \operatorname{ord}_x(f|_{D_1}) \log |k(x)|$. Thus we have

$$\operatorname{ord}_x(f|_{D_1}) = \sum_{i=1}^{r} \operatorname{ord}_{\mathfrak{p}_i}(\varepsilon^* f|_{D_1})[k(\mathfrak{p}_i) : k(x)],$$

where $k(\mathfrak{p}_i)$ denotes the residue field of $\mathfrak{p}_i$. By definition of the non-

Archimedean absolute value, $\|\alpha\|_{\mathfrak{p}_i} = |k(\mathfrak{p}_i)|^{-\operatorname{ord}_{\mathfrak{p}_i}(\alpha)}$. Thus

$$-\sum_{i=1}^{r} \log \|\varepsilon^* f_2|_{D_1}\|_{\mathfrak{p}_i} = \sum_{i=1}^{r} \operatorname{ord}_{\mathfrak{p}_i}(\varepsilon^* f|_{D_1}) \log |k(\mathfrak{p}_i)|$$
$$= \sum_{i=1}^{r} \operatorname{ord}_{\mathfrak{p}_i}(\varepsilon^* f|_{D_1})[k(\mathfrak{p}_i) : k(x)] \log |k(x)|. \quad \square$$

Given two Arakelov divisors D_1 and D_2 that are not contained in distinct fibers of $\mathcal{X} \to B$, their intersection (which we denote $\langle D_1, D_2\rangle$) has a finite and an infinite component. In [17], Arakelov defines the infinite part of the intersection by first defining an “intersection number” for two points P and Q on the Riemann surface $\mathcal{X}_\sigma$ using Green functions (see §4.4). Let $\mathcal{X}_\sigma$ be any Riemann surface with a Hermitian metric μ, and let $P, Q \in \mathcal{X}_\sigma$. Motivated by Lemma 1.53, one might want to define $\langle P, Q\rangle = -\log \phi_P(Q)$, where $\phi_P(z)$ is a function that vanishes to degree one at P (like a uniformizer). However, there are too many functions that satisfy this, so one insists on additional conditions. For example, one requires that ϕ is a non-negative function with a unique zero at P, with a first order zero at P. Imposing further conditions, such as symmetry of $\langle \cdot, \cdot \rangle$ and the normalization $\int \log \phi_P d\mu = 0$, leads to the concept of an *Arakelov–Green function* as defined in §4.4. For more details on the significance of the properties of such functions, we refer the reader to [17]. We now define the total intersection product of two Arakelov divisors.

Definition 1.54 (Intersection of Arakelov divisors)**.** Let D_1 and D_2 be two irreducible Arakelov divisors on an arithmetic surface $\mathcal{X}$. Then the intersection product $\langle D_1, D_2\rangle$ is defined as the symmetric $\mathbf{R}$-bilinear form satisfying the following conditions:

(i) If D_1 is a vertical divisor or a component thereof and D_2 has no components in common with D_1, then $\langle D_1, D_2\rangle := \sum_{b\in B} \langle D_1, D_2\rangle_b$ where the sum is over the closed points of B. This implies that if either D_1 or D_2 is a fiber of $\mathcal{X} \to B$, then there is no infinite component of the intersection.

(ii) Let D_1 be a horizontal divisor and $D_2 = \mathcal{X}_\sigma$ for some σ. Suppose D_1 is the image of a point $\varepsilon : B_F \to \mathcal{X}_K$ for a finite extension F/K. Then $\langle D_1, D_2\rangle$ is defined as the degree $[F : K]$. Equivalently, this is the degree of the residue field of D_1 over K.

(iii) If σ, σ' are two distinct embeddings $K \hookrightarrow \mathbf{C}$, then $\langle \mathcal{X}_\sigma, \mathcal{X}_{\sigma'}\rangle = 0$.

(iv) Suppose D_1 and D_2 are two horizontal sections of $X \to B$, defined over number fields F_1 and F_2 respectively. Fix a $\sigma : K \hookrightarrow \mathbf{C}$. Let $\tau_1 : F_1 \hookrightarrow \mathbf{C}$ and $\tau_2 : F_2 \hookrightarrow \mathbf{C}$ be embeddings that extend σ. These correspond to points P^{τ_1} and P^{τ_2} on the Riemann surface $\mathcal{X}_\sigma$. Define

$$\langle D_1, D_2\rangle_\sigma = \sum_{\tau_1,\tau_2} -\log G_\sigma(P^{\tau_1}, P^{\tau_2}),$$

where G_σ is the Arakelov-Green function attached to $\mathcal{X}_\sigma$.

Finally, the *total intersection* is defined as

$$\langle D_1, D_2\rangle := \sum_{b\in B}\langle D_1, D_2\rangle_b + \sum_{\sigma:K\hookrightarrow\mathbf{C}}\langle D_1, D_2\rangle_\sigma, \tag{5}$$

Proposition 1.55 ([17], Proposition 1.2)**.** *Let D_1 be a principal divisor on $\mathcal{X}$. Then $\langle D_1, D\rangle = 0$ for any divisor D.*

Using Lemma 1.53 and the properties of Arakelov-Green functions, this proposition reduces to the product formula on K.

5.3 Intersections on higher dimensional varieties

We now take a different approach to intersection theory on higher dimensional arithmetic varieties following Fulton [21, Section 2.3], Moriwaki [13], and Gillet-Soulé [19].

Definition 1.56. Let X be any variety and let D be a Cartier divisor on X. Let $j : V \hookrightarrow X$ be a codimension p irreducible subvariety of X. If V is not contained in the support of D, then define

$$D \cdot [V] = [j^*(D)],$$

where $[-]$ is the cycle corresponding to the respective subvariety. If V is contained in the support of D, then $j^*(D)$ is no longer a Cartier divisor on V. In this case, consider the line bundle $j^*\mathcal{O}_X(D)$. Let $[C]$ denote the Weil divisor on V corresponding to the line bundle $\mathcal{O}_V(C)$ that is isomorphic to $j^*\mathcal{O}_X(D)$. Define $D \cdot [V] = [C]$.

Let $Z^p(X)$ denote the set of codimension p cycles on X. Then, extending the above definition linearly, we can define a homomorphism from $Z^p(X) \to Z^{p+1}(X)$. Further, this map respects rational equivalence and thus descends to a homomorphism $\mathrm{CH}^p(X) \to \mathrm{CH}^{p+1}(X)$. When $L = \mathcal{O}_X(D)$, this is precisely the algebraic first Chern Class. We now proceed to define a similar homomorphism in the arithmetic world.

Arithmetic Chow groups

In this subsection, we let $\mathcal{X}$ be an arithmetic variety over $\operatorname{Spec} \mathbf{Z}$ that is regular, with smooth generic fiber. By a metrized line bundle $\overline{\mathcal{L}}$ on $\mathcal{X}$, we will mean a line bundle $\mathcal{L}$ on $\mathcal{X}$ with a metric $\|\cdot\|$ on $\mathcal{L} \otimes \mathbf{C}$ on $\mathcal{X}_\infty(\mathbf{C})$.

Definition 1.57. An *arithmetic cycle* of codimension p is a pair (Z, g), where Z is a codimension p algebraic cycle on $\mathcal{X}$ and g is a Green current for $Z(\mathbf{C})$. An *arithmetic D-cycle* of codimension p is a pair (Z, g) where Z is a codimension p algebraic cycle on $\mathcal{X}$, and g is a current of type $(p-1, p-1)$ on $\mathcal{X}(\mathbf{C})$.

The set of all arithmetic cycles (resp. D-cycles) of codimension p is denoted $\widehat{Z}^p(\mathcal{X})$ (resp. $\widehat{Z}^p_D(\mathcal{X})$). Let $\widehat{R}^p(\mathcal{X})$ denote the subgroup of $\widehat{Z}^p(\mathcal{X})$ generated by:

(i) $(\operatorname{div}(f), [-\log|f|^2])$, where f is a rational function on some subvariety Y of codimension $p-1$, and $[\log|f|^2]$ is the current defined by $\phi \mapsto \int_{Y(\mathbf{C})} \log|f|^2 \wedge \phi$.

(ii) $(0, \partial\alpha + \overline{\partial}\beta)$ where α and β are forms of type $(p-2, p-1)$ and $(p-1, p-2)$ respectively.

Note that $\widehat{R}^p(\mathcal{X})$ can also be considered as subgroup of $\widehat{Z}^p_D(\mathcal{X})$. This allows us to make the following definitions.

Definition 1.58. Define the *arithmetic Chow group* and *arithmetic D-Chow group* of codimension p as $\widehat{\mathrm{CH}}^p(\mathcal{X}) = \widehat{Z}^p(\mathcal{X})/\widehat{R}^p(\mathcal{X})$ and $\widehat{\mathrm{CH}}^p_D(\mathcal{X}) = \widehat{Z}^p_D(\mathcal{X})/\widehat{R}^p(\mathcal{X})$, respectively.

Definition 1.59. Let $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ be a metrized line bundle on $\mathcal{X}$. Let $(Z, g) \in \widehat{Z}^p_D(\mathcal{X})$ and suppose Z is integral. Further, let s be a rational section of $\mathcal{L}|_Z$. Define the map $\tilde{\phi} : \widehat{Z}^p_D(\mathcal{X}) \to \widehat{Z}^{p+1}_D(\mathcal{X})$ by

$$\tilde{\phi} : (Z, g) \mapsto (\operatorname{div}(s), [-\log\|s\|^2_Z] + c_1(\overline{\mathcal{L}}) \wedge g),$$

where $c_1(\overline{\mathcal{L}})$ is as in §4.3. Note that $c_1(\overline{\mathcal{L}})$ is a $(1,1)$-form, so $c_1(\overline{\mathcal{L}}) \wedge g$ is (dual to) a (p,p)-form. Remark 1.43 implies that the form $c_1(\overline{\mathcal{L}})$ vanishes on $\widehat{R}^p(\mathcal{X})$, so the map $\tilde{\phi}$ descends to a homomorphism

$$\widehat{c}_1(\overline{\mathcal{L}}) \cdot (-) : \widehat{\mathrm{CH}}^p_D(\mathcal{X}) \to \widehat{\mathrm{CH}}^{p+1}_D(\mathcal{X}).$$

We call $\widehat{c}_1(\overline{\mathcal{L}})$ the *first arithmetic Chern class*. Just as in the classical case, the first Chern class also admits a description as a cycle. This is given by $\widehat{c}_1(\overline{\mathcal{L}}) = (\operatorname{div}(s), -\log\|s\|^2)$ for some section s of $\mathcal{L}$.

The first Chern class satisfies the following projection formula.

Proposition 1.60 ([13], Proposition 1.2)**.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a projective morphism of generically smooth arithmetic varieties. Let $\overline{\mathcal{L}}$ be a C^∞-Hermitian line bundle on $\mathcal{Y}$, and let $z \in \widehat{\mathrm{CH}}^p_D(\mathcal{X})$. Then $f_*(\widehat{c}_1(f^*\overline{\mathcal{L}}) \cdot z) = \widehat{c}_1(\overline{\mathcal{L}}) \cdot f_*(z)$ (see Definition 1.59).*

Definition 1.61 (Arakelov degree)**.** Let d be the dimension of the generic fiber of $\mathcal{X} \to B$. Define the *arithmetic intersection number* $\widehat{\deg} : \widehat{\mathrm{CH}}^{d+1}_D(\mathcal{X}) \to \mathbf{R}$ by

$$\widehat{\deg}\left(\sum_P n_P P, T\right) = \sum_P n_P \log|k(P)| + \frac{1}{2}\int_{\mathcal{X}(\mathbf{C})} T. \tag{6}$$

An inductive argument using the product formula for number fields implies that $\widehat{\deg}$ is 0 on $\widehat{R}^{d+1}(\mathcal{X})$, so this is well-defined on $\widehat{\mathrm{CH}}^{d+1}_D(\mathcal{X})$. The projection formula for $\widehat{c}_1$ implies a similar projection formula for $\widehat{\deg}$ [13, Proposition 1.3].

6 Moriwaki heights

We now discuss the height function defined by Moriwaki [13]. Moriwaki heights are fairly general and specialize to some of the heights defined in Section 3. Roughly speaking, Moriwaki heights are the higher transcendence degree analogues of geometric heights over number fields (see Definition 1.28). They bridge the gap between geometric heights, which have a pleasant definition but are often poorly behaved, and naïve heights, which are generally well-behaved.

Definition 1.62. Let $\mathcal{B}$ be an arithmetic variety. Let $\mathcal{X}$ be a normal, projective arithmetic variety over $\mathcal{B}$. A metrized line bundle $\overline{\mathcal{L}}$ on $\mathcal{X}$ is *nef* if $c_1(\mathcal{L})$ is semipositive and $\widehat{\deg}(\overline{\mathcal{L}}|_\Gamma) \geq 0$ for all curves $\Gamma \subset \mathcal{B}$.

Definition 1.63. Let K be a finitely generated field over $\mathbf{Q}$, and let $\mathrm{trdeg}_{\mathbf{Q}}(K) = d$. A *polarization* of K is a collection $\overline{\mathcal{B}} = (\mathcal{B}; \overline{\mathcal{H}}_1, \ldots, \overline{\mathcal{H}}_d)$, where $\mathcal{B}$ is a normal, projective, arithmetic variety with fraction field K, and each $\overline{\mathcal{H}}_i$ is a nef C^∞-Hermitian line bundle on $\mathcal{B}$.

Now let $X \to \operatorname{Spec} K$ be a variety with an integral model $\pi : \mathcal{X} \to \mathcal{B}$, that is to say, $\pi : \mathcal{X} \to \mathcal{B}$ is an arithmetic variety whose generic fiber is isomorphic to X. Let L be a metrized $\mathbf{Q}$-line bundle on X that extends to a line bundle $\mathcal{L}$ on $\mathcal{X}$. The pair $(\mathcal{X}, \mathcal{L})$ is called a *model* for (X, L).

Finally, given $P \in X(\overline{K})$, let $\Delta_P \in \mathcal{X}$ be the Zariski closure of the image of P under $X \hookrightarrow \mathcal{X}$. The *Moriwaki height* corresponding to the polarization $\overline{\mathcal{B}}$ is

$$h^{\overline{\mathcal{B}}}_{(\mathcal{X},\mathcal{L})}(P) := \frac{1}{[K(P):K]}\widehat{\deg}\left(\widehat{c}_1(\mathcal{L}|_{\Delta_P}) \cdot \prod_{i=1}^{d} \widehat{c}_1(\pi^*\overline{\mathcal{H}}_i|_{\Delta_P})\right).$$

Up to bounded functions, the Moriwaki height is independent of the choice of model $(\mathcal{X},\mathcal{L})$ [13, Corollary 3.3.5]. One may thus write $h^{\overline{\mathcal{B}}}_{L} := h^{\overline{\mathcal{B}}}_{(\mathcal{X},\mathcal{L})} + O(1)$. By making particular choices for our polarization $\overline{\mathcal{B}}$, we can recover some of the height functions introduced in Section 3.

Example 1.64 (Geometric height)**.** Let K be a number field, so that $\operatorname{trdeg}_{\mathbf{Q}}(K) = 0$. Then $h^{\overline{\mathcal{B}}}_{(\mathcal{X},\mathcal{L})}(P) = \frac{\widehat{\deg}(\widehat{c}_1(\mathcal{L}|_{\Delta_P}))}{[K(P):K]}$, which is the Arakelov-theoretic analog of the geometric height (see Definition 1.28).

Definition 1.65. Let K be a finitely generated extension of $\mathbf{Q}$. Fix a polarization $\overline{\mathcal{B}}$ of K. Let $x := (x_0,\dots,x_n) \in K^{n+1}\backslash\{0\}$. Let Γ be the set of all prime divisors in $\mathcal{B}$. Define the *naïve height* with respect to $\overline{\mathcal{B}}$ as

$$h^{\overline{\mathcal{B}}}_{nv}(x) := \sum_{\gamma\in\Gamma} \max_i\{-\operatorname{ord}_\gamma(x_i)\}\widehat{\deg}\left(\prod_{i=1}^{d}\widehat{c}_1(\overline{\mathcal{H}}_i)|_\gamma\right) \tag{7}$$
$$+ \int_{P\in\mathcal{B}(\mathbf{C})} \log\max_i\{|x_i(P)|\} \bigwedge_{i=1}^{d} c_1(\overline{\mathcal{H}}_i).$$

By [13, Section 3.2], $h^{\overline{\mathcal{B}}}_{nv}$ is well-defined on $\mathbf{P}^n(K)$ and compatible with finite extensions K'/K. The motivation for calling $h^{\overline{\mathcal{B}}}_{nv}$ the naïve height is the following observation.

Remark 1.66. If K is a number field (i.e. $d = 0$), then $\prod_{i=1}^{d}\widehat{c}_1(\overline{\mathcal{H}}_i)|_\gamma = ([\gamma],0)$ and $\bigwedge_{i=1}^{d} c_1(\overline{\mathcal{H}}_i) = 1$. Setting $\mathcal{B} = \operatorname{Spec}\mathcal{O}_K$, we find that $h^{\overline{\mathcal{B}}}_{nv}$ recovers the naïve logarithmic height (Definition 1.13):

$$\begin{aligned} h^{\overline{\mathcal{B}}}_{nv}(x) &= \sum_{\mathfrak{p}\nmid\infty} \log|k(\mathfrak{p})| \cdot \max_i\{-\operatorname{ord}_{\mathfrak{p}}(x_i)\} + \sum_{\mathfrak{p}|\infty}\log\max_i\{|x_i|_{\mathfrak{p}}\} \\ &= \log h_K(x). \end{aligned}$$

Similarly, $h^{\overline{\mathcal{B}}}_{nv}$ is induced by a generalized global field structure on $\mathbf{Q}(t_1,\dots,t_d)$.

Theorem 1.67. *Let $K = \mathbf{Q}(\mathbf{P}^d)$. Let $\mathcal{B} = \mathbf{P}^d_{\mathbf{Z}}$ and $\overline{\mathcal{H}}_i = (\mathcal{O}_{\mathcal{B}}(1), \|\cdot\|_\infty)$ for $1 \le i \le d$. Then there exists a generalized global field structure on K such that $h^{\overline{\mathcal{B}}}_{nv}$ is the logarithmic standard height.*

Proof Let Γ be the set of Weil divisors $\gamma \subset \mathcal{B}$, and let $\Gamma_\infty = \Gamma \cup \{\infty\}$. For each $\gamma \in \Gamma$, set $\|x\|_\gamma = e^{-\lambda_\gamma \operatorname{ord}_\gamma(x)}$, where $\lambda_\gamma = \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{O}_{\mathcal{B}}(1)}|_\gamma)^d)$. Define the absolute value at the Archimedean place to be $\|x\|_\infty$ (see Example 1.37). Then $(K, \{\|\cdot\|_v\}_{v\in\Gamma_\infty})$ is a generalized global field, and $h^{\overline{\mathcal{B}}}_{nv}$ is the corresponding logarithmic standard height:

$$\begin{aligned} h^{\overline{\mathcal{B}}}_{nv}(x) &= \sum_{\gamma\in\Gamma} \log \max_i\{\|x_i\|_\gamma\} + \int_{\mathcal{B}(\mathbf{C})} \log \max_i\{\|x_i\|_\infty\} \\ &= \sum_{v\in\Gamma_\infty} e_v \log \max_i\{\|x_i\|_v\}. \end{aligned}$$

(Note that the Archimedean place is real, so $e_v = 1$ for all $v \in \Gamma_\infty$.) To see that $(K, \{\|\cdot\|_v\}_{v\in\Gamma_\infty})$ is indeed a generalized global field, it suffices to verify the generalized product formula for $\{\|\cdot\|_v\}$. Given $x \in K^\times$, this is equivalent to computing $\sum_{v\in\Gamma_\infty} e_v \log\|x\|_v = 0$. Indeed, note that

$$\begin{aligned} \sum_{v\in\Gamma_\infty} e_v \log\|x\|_v &= \sum_{\gamma\in\Gamma}(-\operatorname{ord}_\gamma(x))\lambda_\gamma + \int_{\mathcal{B}(\mathbf{C})} \log\|x\|_\infty \bigwedge_{i=1}^{d} c_1(\overline{\mathcal{O}_{\mathcal{B}}(1)}) \\ &= \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{O}_{\mathcal{B}}(1)})^d \cdot \widehat{(x^{-1})}), \end{aligned}$$

which is equal to 0 since $\widehat{(x^{-1})}$ is a principal divisor. □

By making a particular choice of (X, L), we recover $h^{\overline{\mathcal{B}}}_{nv}$ from $h^{\overline{\mathcal{B}}}_{(\mathcal{X},\mathcal{L})}$ [13, Proposition 3.3.2].

Proposition 1.68. *Let $X = \mathbf{P}^n_K$ and $L = (\mathcal{O}_X(1), \|\cdot\|_\infty)$. Then $h^{\overline{\mathcal{B}}}_{(\mathcal{X},\mathcal{L})} = h^{\overline{\mathcal{B}}}_{nv}$.*

Proof Let $(\mathcal{X}, \mathcal{L}) = (\mathbf{P}^n_{\mathcal{B}}, \mathcal{O}_{\mathbf{P}^n_{\mathcal{B}}}(1))$. Here, $\mathbf{P}^n_{\mathcal{B}} = \mathbf{P}^n_{\mathbf{Z}} \times_{\mathbf{Z}} \mathcal{B}$ has a projection to $\mathbf{P}^n_{\mathbf{Z}}$, and $(\mathcal{O}_{\mathbf{P}^n_{\mathcal{B}}}(1), \|\cdot\|_\infty)$ are defined to be the pullback of $(\mathcal{O}_{\mathbf{P}^n_{\mathbf{Z}}}(1), \|\cdot\|_\infty)$. We prove the case $d = 1$ for simplicity. The case of general d can be proved similarly. Further, for simplicity of notation, suppose that P is defined over the field K. Thus

$$h^{\overline{\mathcal{B}}}_{(\mathcal{X},\mathcal{L})}(P) = \widehat{\deg}(\widehat{c}_1(\pi^*\overline{\mathcal{H}}|_{\Delta_P}) \cdot \widehat{c}_1(\mathcal{L}|_{\Delta_P})),$$

where $\pi : \mathbf{P}^n_{\mathcal{B}} \to \mathcal{B}$ is the structure map and $\overline{\mathcal{B}} = (\mathcal{B}, \overline{\mathcal{H}})$ is a polarization of $\mathcal{B}$. Since Δ_P is the closure of a map $\operatorname{Spec} K \to X$, the properness of X gives us an induced map $s_P : \mathcal{B} \to \Delta_P \hookrightarrow \mathbf{P}^n_{\mathcal{B}}$. Let x denote a section of $\mathcal{L}$ such that $s_P^*(x) \neq 0$. Since $\pi|_{\Delta_P}$ is generically of degree 1, Proposition 1.60 implies

$$\begin{aligned}\pi_*(\widehat{c}_1(\pi^*\overline{\mathcal{H}}|_{\Delta_P}) \cdot \widehat{c}_1(\mathcal{L}|_{\Delta_P})) &= \widehat{c}_1(\overline{\mathcal{H}}) \cdot \pi_*(\widehat{c}_1(\mathcal{L}|_{\Delta_P})) \\ &= \widehat{c}_1(\overline{\mathcal{H}}) \cdot \mathrm{div}(s_P^*(x))).\end{aligned}$$

Thus

$$h^{\overline{\mathcal{B}}}_{(\mathcal{X},\mathcal{L})} = \widehat{\deg}(\widehat{c}_1(H) \cdot \mathrm{div}(s_P^*(x))). \tag{8}$$

Write $\mathrm{div}(s_P^*(x)) = \sum_{\gamma \subset B} a_\gamma \gamma$, where $a_\gamma \in \mathbf{Z}$ and the sum is over irreducible divisors $(\gamma, g_\gamma) \subset B$.[1] The finite contribution to the Arakelov height comes from the cycle $\sum_\gamma a_\gamma(\gamma \cdot \overline{\mathcal{H}})$, we conflate the line bundle $\overline{\mathcal{H}}$ with the Weil divisor corresponding to it and the intersection is as in Definition 1.56. Write $\gamma \cdot H = \sum_i Q_i^\gamma$ as a sum of points. By definition,

$$\widehat{c}_1(\overline{\mathcal{H}}) \cdot \mathrm{div}(s_P^*(x)) = \left(\sum_{\gamma,i} a_\gamma Q_i^\gamma, [-\log \|s_P^*(x)|_{\overline{\mathcal{H}}}\|_\infty^2] + \sum_\gamma c_1(\overline{\mathcal{H}}) \wedge g_\gamma\right).$$

Equation 8 thus implies

$$\begin{aligned}&\widehat{\deg}(\widehat{c}_1(\pi^*\overline{\mathcal{H}}|_{\Delta_P}) \cdot \widehat{c}_1(\mathcal{L}|_{\Delta_P})) \\ &= \sum_{\gamma,i} a_\gamma \log |k(Q_i^\gamma)| + \int_{B(\mathbf{C})} -\log \|s_P^*(x)|_{\overline{\mathcal{H}}}\|_\infty + \sum_\gamma \int_{B(\mathbf{C})} c_1(\overline{\mathcal{H}}) \wedge g_\gamma \\ &= \sum_\gamma a_\gamma \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{H}}|_\gamma)) + \int_{B(\mathbf{C})} -\log \|s_P^*(x)\|_\infty c_1(\overline{\mathcal{H}}).\end{aligned}$$

Now let $P = [p_0 : \ldots : p_n]$ and suppose (without loss of generality) that $p_0 \neq 0$. Then x_0 is a non-vanishing section of $\mathcal{O}_{\mathbf{P}^n_{\mathcal{B}}}(1)$ around P. By Example 1.37, we have that $\|s_P^*(x_0)\|_\infty = \frac{|p_0|}{\max\{|p_0|,\ldots,|p_n|\}}$. Thus $-\log \|s_P^*(x_0)\|_\infty = \log \max_i\{|p_i|\} - \log |p_0|$.

For the finite places, note that the sections $x_0, \ldots, x_n$ generate $\mathcal{O}_{\mathbf{P}^n_{\mathcal{B}}}(1)$ and thus $s_P^*(x_0), \ldots, s_P^*(x_n)$ globally generate $s_P^*(\mathcal{O}_{\mathbf{P}^n_{\mathcal{B}}}(1))$. Since $s_P : \mathcal{B} \hookrightarrow \mathbf{P}^n_{\mathcal{B}}$ is an embedding, we have

$$\begin{aligned}a_\gamma = \mathrm{ord}_\gamma(s_P^*(x_0)) &= \mathrm{len}_{\mathcal{O}_{\mathcal{B},\gamma}}(\mathcal{O}_{\mathcal{B},\gamma}/s_P^* x_0) \\ &= \mathrm{len}_{\mathcal{O}_{\mathcal{B},\gamma}}(s_P^*\mathcal{O}(1)_\gamma/s_P^* x_0) \\ &= \mathrm{len}_{\mathcal{O}_{\mathcal{B},\gamma}}\left(\frac{\mathcal{O}_{\mathcal{B}} p_0 + \ldots + \mathcal{O}_{\mathcal{B}} p_n}{\mathcal{O}_{\mathcal{B}} p_0}\right) \\ &= \max_i\{-\mathrm{ord}_\gamma(p_i)\} + \mathrm{ord}_\gamma(p_0).\end{aligned}$$

□

[1] Here, $g_\gamma = -\log \|t_\gamma\|^2$ for some section t_γ of $\mathcal{O}(\gamma)$.

Remark 1.69 (Northcott property)**.** Moriwaki heights need not satisfy the Northcott property in general. However, if $\overline{\mathcal{H}}_i \to \mathcal{B}$ is ample for $1 \leq i \leq d$, then $h^{\overline{B}}_{(\mathcal{X},\mathcal{L})}$ is Northcott [13, Proposition 3.3.7 (4)] (the positivity assumptions *nef* and *big* are both implied by *ample*).

Acknowledgements We thank the organizers of the Stacks Project Workshop and Max Lieblich for their support and guidance. We thank Will Sawin for explaining geometric heights to us. We would also like to thank the referees for their detailed and helpful comments on the early draft of this article. The second named author also thanks Brandon Alberts for many helpful conversations. The first named author received support from Kirsten Wickelgren's NSF CAREER grant (DMS-1552730).

References

[1] Jens Franke, Yuri I. Manin, and Yuri Tschinkel. Rational points of bounded height on fano varieties. *Inventiones mathematicae*, 95(2):421–435, 1989.

[2] Emmanuel Peyre. Hauteurs et mesures de tamagawa sur les variétés de fano. *Duke Mathematical Journal*, 79(1):101–218, 1995.

[3] V. V. Batyrev and Yu. I. Manin. Sur le nombre des points rationnels de hauteur borné des variétés algébriques. *Mathematische Annalen*, 286 (1-3):27–43, 1990.

[4] Paul Vojta. *Diophantine approximations and value distribution theory*, volume 1239 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1987. ISBN 3-540-17551-2.

[5] Emmanuel Ullmo. Positivité et discrétion des points algébriques des courbes. *Annals of Mathematics (2)*, 147(1):167–179, 1998.

[6] Shou-Wu Zhang. Equidistribution of small points on abelian varieties. *Annals of Mathematics (2)*, 147(1):159–165, 1998.

[7] André Weil. L'arithmétique sur les courbes algébriques. *Acta Math.*, 52 (1):281–315, 1929. ISSN 0001-5962. doi: 10.1007/BF02547409. URL `https://doi.org/10.1007/BF02547409`.

[8] G. Faltings. Endlichkeitssätze für abelsche Varietäten über Zahlkörpern. *Invent. Math.*, 73(3):349–366, 1983. ISSN 0020-9910. doi: 10.1007/BF01388432. URL `https://doi.org/10.1007/BF01388432`.

[9] Paul Vojta. Siegel's theorem in the compact case. *Ann. of Math. (2)*, 133(3):509–548, 1991. ISSN 0003-486X. doi: 10.2307/2944318. URL `https://doi.org/10.2307/2944318`.

[10] Antoine Chambert-Loir. Heights and measures on analytic spaces. A survey of recent results, and some remarks. In *Motivic integration and its interactions with model theory and non-Archimedean geometry. Volume*

II, volume 384 of *London Math. Soc. Lecture Note Ser.*, pages 1–50. Cambridge Univ. Press, Cambridge, 2011.

[11] Antoine Chambert-Loir. Chapter VII: Arakelov geometry, heights, equidistribution, and the Bogomolov conjecture. In *Arakelov geometry and Diophantine applications*, volume 2276 of *Lecture Notes in Math.*, pages 299–328. Springer, Cham, 2021. doi: 10.1007/978-3-030-57559-5_8. URL https://doi.org/10.1007/978-3-030-57559-5_8.

[12] Brian Conrad. Chow's K/k-image and K/k-trace, and the Lang-Néron theorem. *Enseign. Math. (2)*, 52(1-2):37–108, 2006. ISSN 0013-8584.

[13] Atsushi Moriwaki. Arithmetic height functions over finitely generated fields. *Inventiones mathematicae*, 140:101–142, 2000. doi: https://doi.org/10.1007/s002220050358.

[14] Jan-Hendrik Evertse and Kálmán Győry. *Unit equations in Diophantine number theory*, volume 146 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2015. ISBN 978-1-107-09760-5. doi: 10.1017/CBO9781316160749. URL https://doi.org/10.1017/CBO9781316160749.

[15] Xinyi Yuan and Shou-Wu Zhang. Adelic line bundles over quasi-projective varieties, 2021.

[16] Serge Lang. *Diophantine geometry*. Interscience Tracts in Pure and Applied Mathematics, No. 11. Interscience Publishers (a division of John Wiley & Sons), New York-London, 1962.

[17] S. Yu. Arakelov. An intersection theory for divisors on an arithmetic surface. *Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya*, 38:1179—-1192, 1974. URL http://dx.doi.org/10.1070/IM1974v008n06ABEH002141.

[18] Marc Hindry and Joseph H. Silverman. *Diophantine geometry*, volume 201 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000. ISBN 0-387-98975-7; 0-387-98981-1. doi: 10.1007/978-1-4612-1210-2. URL https://doi.org/10.1007/978-1-4612-1210-2. An introduction.

[19] Henri Gillet and Christophe Soulé. Arithmetic intersection theory. *Publications Mathématiques de l'IHÉS*, 72:93–174, 1990. URL http://www.numdam.org/item/PMIHES_1990__72__93_0/.

[20] J. S. Milne. Algebraic number theory. *www.jmilne.org/math*, 2009.

[21] William Fulton. *Intersection Theory*. Springer-Verlag New York, 2nd edition, 1998. ISBN 978-0-387-98549-7. doi: 10.1007/978-1-4612-1700-8.

[22] M. F. Becker and S. MacLane. The minimum number of generators for inseparable algebraic extensions. *Bull. Amer. Math. Soc.*, 46:182–186, 1940. ISSN 0002-9904. doi: 10.1090/S0002-9904-1940-07169-1. URL https://doi.org/10.1090/S0002-9904-1940-07169-1.

[23] Jean-Pierre Serre. *Local fields*, volume 67 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1979. ISBN 0-387-90424-7. Translated from the French by Marvin Jay Greenberg.

[24] Michael D. Fried and Moshe Jarden. *Field arithmetic*, volume 11 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of*

Modern Surveys in Mathematics. Springer-Verlag, Berlin, third edition, 2008. Revised by Jarden.

[25] Jean-Pierre Serre. *Lectures on the Mordell-Weil theorem*. Aspects of Mathematics, E15. Friedr. Vieweg & Sohn, Braunschweig, 1989. ISBN 3-528-08968-7. doi: 10.1007/978-3-663-14060-3. URL https://doi.org/10.1007/978-3-663-14060-3. Translated from the French and edited by Martin Brown from notes by Michel Waldschmidt.

[26] The Stacks Project Authors. *Stacks Project*. https://stacks.math.columbia.edu, 2021.

[27] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II. *Inst. Hautes Études Sci. Publ. Math.*, (24):231, 1965. ISSN 0073-8301. URL http://www.numdam.org/item?id=PMIHES_1965__24__231_0.

[28] Gerd Faltings. Calculus on arithmetic surfaces. *Annals of Mathematics*, 119(2):387–424, 1984. ISSN 0003486X. URL http://www.jstor.org/stable/2007043.

[29] Daniel Huybrechts. *Complex geometry*. Universitext. Springer-Verlag, Berlin, 2005. ISBN 3-540-21290-6. doi: 10.1007/b137952. URL https://doi.org/10.1007/b137952. An introduction.

[30] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry*. Pure and Applied Mathematics. John Wiley and Sons, New York, 1978. ISBN 0-471-32792-1.

[31] Serge Lang. *Introduction to Arakelov theory*. Springer-Verlag, New York, 1988. ISBN 0-387-96793-1. doi: 10.1007/978-1-4612-1031-3. URL https://doi.org/10.1007/978-1-4612-1031-3.

[32] Joseph H. Silverman. *Advanced topics in the arithmetic of elliptic curves*. Graduate Texts in Mathematics, 151. Springer-Verlag, New York, 1994. ISBN 0-387-94328-5.

[33] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron Models*, volume 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*. Springer-Verlag, Berlin, 1990. ISBN 3-540-50587-3.

[34] Qing Liu. *Algebraic geometry and arithmetic curves*, volume 6 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2002. ISBN 0-19-850284-2. Translated from the French by Reinie Erné, Oxford Science Publications.