KSP-CHARACTERISTIC CLASSES DETERMINE SPIN^h COBORDISM

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Abstract. A classic result of Anderson, Brown, and Peterson states that the cobordism spectrum MSpin (respectively, MSpin^c) splits as a sum of Eilenberg–Mac Lane spectra and connective covers of real K-theory (respectively, complex K-theory) at 2. We develop a theory of symplectic K-theory classes and use these to build an explicit splitting for MSpin^h in terms of Eilenberg–Mac Lane spectra and spectra related to symplectic K-theory. This allows us to determine the Spin^h cobordism groups systematically. We also prove that two Spin^h-manifolds are cobordant if and only if their underlying unoriented manifolds are cobordant and their KSp-characteristic numbers agree.

Contents

1. Introduction

There is an intimate connection, brought to the fore by Atiyah–Bott–Shapiro [ABS64], between topological K-theory and spin geometry. This connection was further strengthened in the work of Hopkins–Hovey [HH92]. A crucial bridge between these two results

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is built in the work of Anderson–Brown–Peterson, who gave a 2-local splitting of the cobordism spectra MSpin and MSpin^c [ABP67]. The Anderson–Brown–Peterson splitting of MSpin and MSpin^c also yields combinatorial formulas for the Spin and Spin^c cobordism groups.

The goal of this work is to give an explicit splitting for the cobordism spectrum MSpin^h (analogous to the Anderson–Brown–Peterson splittings of MSpin and MSpin^c) in terms of ordinary cohomology classes and KSp-characteristic classes. Here, Spin^h is the *quaternionic spin group*, defined as the colimit of the double covers $\mathrm{Spin}^h(n)$ of $\mathrm{SO}(n)\times\mathrm{Sp}(1)$. Quaternionic spin theory was first studied systematically by Nagase [Nag95] and subsequently by Okonek–Teleman [OT96] and Bär [Bär99], although $\mathrm{Spin}^h(4)$ appeared even earlier [BFF78, HP78]. There has been a recent resurgence of interest in quaternionic spin theory, in part due to its role in physics [Che17, SSR17, LS19, FH21, AM21, Law23, Hu23].

Let $\mathcal{P}_{\mathrm{even}}$ and $\mathcal{P}_{\mathrm{odd}}$ denote the sets of even and odd partitions, respectively. Our main result is an explicit analog of the Anderson–Brown–Peterson splitting.

Theorem 1.1. *Let F be the fiber of the map $\mathrm{ko}\to H\mathbb{Z}/2\mathbb{Z}$ classifying the non-trivial element of $H^0(\mathrm{ko};\mathbb{Z}/2\mathbb{Z})$. Then there are cohomology classes $Z\subset H^*(\mathrm{MSpin}^h;\mathbb{Z}/2\mathbb{Z})$ and a map of spectra*

$$\mathrm{MSpin}^h\to\bigvee_{I\in\mathcal{P}_{\mathrm{even}}}\mathrm{ksp}\langle 4|I|\rangle\vee\bigvee_{I\in\mathcal{P}_{\mathrm{odd}}}\Sigma^{4|I|}F\vee\bigvee_{z\in Z}\Sigma^{\deg z}H\mathbb{Z}/2\mathbb{Z}$$

that is a 2-local equivalence.

We prove Theorem 1.1 by studying the mod 2 cohomology and homotopy groups of each summand, as well as describing the behavior of the map from MSpin^h to each summand in cohomology. We then show that the map

$$\mathrm{MSpin}^h\to\bigvee_{I\in\mathcal{P}_{\mathrm{even}}}\mathrm{ksp}\langle 4|I|\rangle\vee\bigvee_{I\in\mathcal{P}_{\mathrm{odd}}}\Sigma^{4|I|}F$$

induces an isomorphism on certain associated Margolis homology groups. We conclude by taking the cokernel of the induced map on cohomology to construct the necessary Eilenberg–Mac Lane summands.

A key input to our approach is the construction of characteristic classes

$$\kappa^I\in\mathrm{ksp}\langle 4|I|\rangle^0(\mathrm{MSpin}^h),$$
$$\varepsilon^I\in\Sigma^{4|I|}F^0(\mathrm{MSpin}^h),$$

which we call KSp-*Pontryagin classes* and *elephant classes*, respectively. These have associated KSp-*characteristic numbers*, which can be used to detect cobordisms between Spin^h-manifolds.

Theorem 1.2. *Two Spin^h-manifolds are cobordant if and only if their* KSp*-characteristic numbers and $\mathbb{Z}/2\mathbb{Z}$-characteristic numbers are equal.*

We also discuss the asymptotic growth of Spin^h cobordism groups, explicitly calculate the cobordism groups through degree 19999 (and provide the code used in this calculation), compute a KSp-characteristic number of the Wu manifold, and list a few problems of interest in Spin^h geometry.

Remark 1.3. During the preparation of this article, Mills released independent work that obtains some of the same results as us [Mil23]. In *loc. cit.* and this paper, we both derive a splitting at 2 of MSpin^h and use it to calculate Spin^h cobordism groups. However, in *loc. cit.*, the splitting is derived formally from the cohomology of MSpin^h, while our splitting is constructed explicitly from KSp-Pontryagin classes and the quaternionic Atiyah–Bott–Shapiro map $\varphi^h : \mathrm{MSpin}^h \to \mathrm{ksp}$. As a result of this explicit approach, Theorem 1.1 is a strengthening of [Mil23, Theorem 1.1].

1.1. **Outline.** The layout of our article is as follows.

- In Section 2 we summarize basic facts and constructions involving KSp and MSpin^h.
- We give an overview of Anderson, Brown, and Peterson's approach to splitting MSpin in Section 3. We then discuss how this inspires our approach to splitting MSpin^h.
- In Sections 4 and 5, we explore the cohomology of relevant spaces and spectra and discuss the maps of the splitting in cohomology.
- In Section 6 we study the Margolis homology of the relevant Steenrod modules and show that the map from MSpin^h to the sum of the $\mathrm{ksp}\langle 4|I|\rangle$ and $\Sigma^{4|I|}F$ is an isomorphism on Margolis homology.
- In Section 7, we define the ordinary cohomology classes involved in the splitting. We then prove Theorem 1.1 using the isomorphism on Margolis homology and a filtering procedure. This filtering procedure is inspired by one used in [ABP67], although some modifications are necessary due to MSpin^h not being a ring spectrum.
- We discuss the computation of Spin^h cobordism groups in Section 8, as well as their asymptotic growth. Tables 2, 3, and 4 allow the reader to compare the Spin, Spin^c, and Spin^h cobordism groups through degree 99.
- In Section 9 we define the KSp-characteristic numbers of a Spin^h manifold and prove Theorem 1.2.
- We outline some potential applications and related questions in Section 10.

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Table 1. Bott periodicity in topological K-theory

$n \pmod 8$	0	1	2	3	4	5	6	7
$\pi_n\mathrm{KU}$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
$\pi_n\mathrm{KO}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0
$\pi_n\mathrm{KSp}$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0

2. Quick facts about KSp and MSpin^h

In this section, we will recall some relevant background material. To begin, we will discuss symplectic K-theory. We will then give a brief introduction to Spin^h geometry and gather some useful results from throughout the literature. See [Law23] for a nice survey of recent developments on Spin^h manifolds.

2.1. KSp. Topologically, Bott periodicity manifests as a repeating pattern in the loop spaces $\Omega^n\mathrm{BO}$, $\Omega^n\mathrm{BU}$, and $\Omega^n\mathrm{BSp}$. One can then define the K-theory spectra KO, KU, and KSp as the Ω-spectra associated to BO, BU, and BSp, respectively. It follows that these topological K-theory groups will repeat periodically (see Table 1).

In the process of proving Bott periodicity for BO, one encounters the homotopy equivalences $\Omega^4\mathrm{BO} \simeq \mathrm{BSp} \times \mathbb{Z}$ and $\Omega^4\mathrm{BSp} \simeq \mathrm{BO} \times \mathbb{Z}$ (which are visible in Table 1). This means that we get a homotopy equivlence of Ω-spectra $\Sigma^4\mathrm{KO} \to \mathrm{KSp}$, which is simply the identity map in each degree. In fact, the equivalence $\Sigma^4\mathrm{KO} \simeq \mathrm{KSp}$ is more than just an equivalence of spectra: it is an equivalence of KO-modules.

Proposition 2.1. *The homotopy equivalence* $\Sigma^4\mathrm{KO} \simeq \mathrm{KSp}$ *is an equivalence of* KO-*modules.*

Proof. This is a standard fact, but we will point to a reference for the reader's convenience. The KO-module structure on KSp is induced by taking the tensor product of a quaternionic bundle with a real bundle, which yields a quaternionic bundle. One has to show that this module map is a degree 4 shift of the tensor product of two real bundles, since the KO-module structure on $\Sigma^4\mathrm{KO}$ is given by

$$\mathrm{KO} \wedge \Sigma^4\mathrm{KO} \simeq \Sigma^4(\mathrm{KO} \wedge \mathrm{KO}) \xrightarrow{\Sigma^4\mu} \mathrm{KO}.$$

(Here, $\mu : \mathrm{KO} \wedge \mathrm{KO} \to \mathrm{KO}$ is the ring structure induced by the tensor product of real bundles.) That the KO-module map on KSp is indeed a degree 4 shift of the ring map on KO is worked out in [Str92, §7]. The relevant quaternionic bundle is denoted by θ in *loc. cit.* □

2.2. $\mathrm{Spin}^h(n)$. We begin by introducing the Spin^h groups. Write $\{\pm 1\}$ to denote the matrix group consisting of the identity matrix and its negative. Recall that $\mathrm{Spin}(n)$ is the universal cover of $\mathrm{SO}(n)$ for $n \geq 3$. Since $\mathrm{Spin}(n) \to \mathrm{SO}(n)$ is a double cover, we get a short exact sequence

$$1 \to \{\pm 1\} \to \mathrm{Spin}(n) \to \mathrm{SO}(n) \to 1.$$

Analogously, $\mathrm{Spin}^c(n)$ is defined as the double (not universal) cover of $\mathrm{SO}(n) \times \mathrm{U}(1)$, giving us the exact sequence

$$1 \to \{\pm 1\} \to \mathrm{Spin}^c(n) \to \mathrm{SO}(n) \times \mathrm{U}(1) \to 1.$$

We may thus realize $\mathrm{Spin}^c(n)$ as the quotient $(\mathrm{Spin}(n) \times \mathrm{U}(1))/\{\pm 1\} \cong \mathrm{Spin}(n) \times_{\{\pm 1\}} \mathrm{U}(1)$. Regarding the unitary factor in $\mathrm{Spin}^c(n)$ as carrying complex structure, we are inclined to rewrite $\mathrm{Spin}(n)$ as $\mathrm{Spin}(n) \cong \mathrm{Spin}(n) \times_{\{\pm 1\}} \mathrm{O}(1)$. This indicates how quaternionic (i.e. symplectic) structure should be introduced.

Definition 2.2. Let $n \geq 3$. The *quaternionic spin group* $\mathrm{Spin}^h(n)$ is defined to be the double cover of $\mathrm{SO}(n) \times \mathrm{SO}(3)$. Equivalently, define

$$\mathrm{Spin}^h(n) := \mathrm{Spin}(n) \times_{\{\pm 1\}} \mathrm{Sp}(1).$$

Remark 2.3. The universal cover of $\mathrm{SO}(n, \mathbb{C})$ is often called complex spin, but this is different from Spin^c. We will never work with $\mathrm{SO}(n, \mathbb{C})$ in this article, so by *complex spin* we always mean Spin^c.

There is a commutative diagram

$$\begin{tikzcd}
\mathrm{Spin}(n) \arrow[r] \arrow[dr] & \mathrm{Spin}^c(n) \arrow[r] \arrow[d] & \mathrm{Spin}^h(n) \arrow[dl] \\
& \mathrm{SO}(n) &
\end{tikzcd}$$

The map $\mathrm{Spin}(n) \to \mathrm{Spin}^c(n)$ is the composition of the inclusion of $\mathrm{Spin}(n)$ into $\mathrm{Spin}(n) \times U(1)$ followed by the quotient map $\mathrm{Spin}(n) \times U(1) \to \mathrm{Spin}^c(n)$, and the map $\mathrm{Spin}^c(n) \to \mathrm{Spin}^h(n)$ is induced by the inclusion $U(1) \to \mathrm{Sp}(1)$ and passage to quotients. The maps $\mathrm{Spin}(n) \times_{\{\pm 1\}} G \to \mathrm{SO}(n)$ are induced by the composition of the projection $\mathrm{Spin}(n) \times_{\{\pm 1\}} G \to \mathrm{Spin}(n)$ and the double cover $\mathrm{Spin}(n) \to \mathrm{SO}(n)$ and passage to the quotient group.

We now recall the definition of a Spin^h structure, which was first introduced by Nagase [Nag95, p. 94].

Definition 2.4. A *Spin^h structure* on a principal $\mathrm{SO}(n)$-bundle $P_{\mathrm{SO}(n)}$ consists of

(i) a principal $\mathrm{SO}(3)$-bundle $P_{\mathrm{SO}(3)}$,

(ii) a principal $\mathrm{Spin}^h(n)$-bundle $P_{\mathrm{Spin}^h(n)}$,

(iii) and a double cover $P_{\mathrm{Spin}^h(n)} \to P_{\mathrm{SO}(n)} \times P_{\mathrm{SO}(3)}$ that is equivariant with respect to $\mathrm{Spin}^h(n) \to \mathrm{SO}(n) \times \mathrm{SO}(3)$.

A *Spin^h manifold* is a manifold whose tangent bundle admits a Spin^h structure.

2.3. **Spin^h-cobordism.** Now that we have a sequence of topological groups $\mathrm{Spin}^h(n)$, we can speak of cobordisms of manifolds with stable Spin^h structure. The resulting cobordism groups are encoded as the homotopy groups of the Spin^h-cobordism spectrum. Spin^h-cobordism and the quaternionic Atiyah–Bott–Shapiro map were developed independently by Hu [Hu22] and the seminal work of Freed and Hopkins on invertible topological phases [FH21].

Definition 2.5. Let BSpin^h be the classifying space of stable Spin^h-vector bundles. Then the Spin^h-*cobordism spectrum* is the Thom spectrum MSpin^h, whose n^{th} space is the Thom space of the universal bundle over $\mathrm{BSpin}^h(n)$.

The maps between the Spin, Spin^c, and Spin^h groups induce a homotopy commutative diagram of classifying spaces

$$\begin{tikzcd}
\mathrm{BSpin} \arrow[r] \arrow[dr] & \mathrm{BSpin}^c \arrow[r] \arrow[d] & \mathrm{BSpin}^h \arrow[dl] \\
& \mathrm{BSO} &
\end{tikzcd}$$

and therefore a diagram of Thom spectra

$$\mathrm{MSpin} \longrightarrow \mathrm{MSpin}^c \longrightarrow \mathrm{MSpin}^h.$$

In contrast to MSpin and MSpin^c, the spectrum MSpin^h does not admit a ring structure. This comes from the fact that there is no "quaternionic tensor product" of vector spaces. That is, the tensor product of two quaternionic vector spaces need not be quaternionic, so the product of two Spin^h manifolds need not be Spin^h. However, the tensor product of a real vector space and a quaternionic vector space is again quaternionic, which suggests that MSpin^h might be an MSpin-module. This was proved by Freed–Hopkins using an explicit *shearing map* [FH21, Equation (10.20)], but we will recall the relevant details.

Setup 2.6. Note that the data of a $\mathrm{Spin}^h(n)$-bundle is equivalent to a pair (E_n, E_3), where E_n is a principal $\mathrm{SO}(n)$-bundle and E_3 is a principal $\mathrm{SO}(3)$-bundle such that $w_2(E_n) = w_2(E_3)$, where w_i denotes the i^{th} mod 2 Stiefel–Whitney class. Recall that $w_1(P) = w_2(P) = 0$ for any principal $\mathrm{Spin}(n)$-bundle P. Indeed, w_1 vanishes on all $\mathrm{SO}(n)$-bundles. For w_2, the short exact sequence $1 \to \mathbb{Z}/2 \to \mathrm{Spin}(n) \to \mathrm{SO}(n) \to 1$ induces an exact sequence on cohomology

$$H^1(-;\mathrm{Spin}(n)) \to H^1(-;\mathrm{SO}(n)) \xrightarrow{w_2} H^2(-;\mathbb{Z}/2),$$

so an $\mathrm{SO}(n)$-bundle lifts to a $\mathrm{Spin}(n)$-bundle if and only if w_2 vanishes. It follows that $(P \oplus E_3, E_3)$ corresponds to a $\mathrm{Spin}^h(n+3)$-bundle, since $P \oplus E_3$ is a principal $\mathrm{SO}(n+3)$-bundle and

$$w_2(P \oplus E_3) = w_2(P) + w_1(P)w_1(E_3) + w_2(E_3) = w_2(E_3).$$

This gives us the shearing map on classifying spaces:

$$\begin{aligned} \mathrm{BSpin}(n) \times \mathrm{BSO}(3) &\to \mathrm{BSpin}^h(n+3) \\ (P, E_3) &\mapsto (P \oplus E_3, E_3). \end{aligned}$$

Applying (homotopy) colimits, we get a map $\mathrm{BSpin} \times \mathrm{BSO}(3) \to \mathrm{BSpin}^h$. This map admits a homotopy inverse $(R, E_3) \mapsto (R \oplus (-E_3), E_3)$, where $-E_3$ is the virtual bundle associated to E_3 (which exists since we are working stably).

Lemma 2.7 (Freed–Hopkins). *The map* $\mathrm{BSpin}(n) \times \mathrm{BSO}(3) \to \mathrm{BSpin}^h(n+3)$ *of classifying spaces over* BO *given in Setup 2.6 induces a homotopy equivalence* $\Sigma^{-3}\mathrm{MSpin} \wedge \mathrm{MSO}(3) \to \mathrm{MSpin}^h$.

Proof. Because $\mathrm{BSpin} \times \mathrm{BSO}(3) \to \mathrm{BSpin}^h$ is a homotopy equivalence, the result follows by taking Thom spectra. The shift by -3 can be seen at the level of Thom spaces, since the Thom space $\mathrm{MSpin}(n) \wedge \mathrm{MSO}(3)$ maps to $\mathrm{MSpin}^h(n+3)$. ☐

Anderson–Brown–Peterson prove a 2-local splitting of the Thom spectra MSpin and MSpin^c [ABP67]. Since the homotopy groups of MSpin and MSpin^c have no odd torsion [Sto68, p. 336], it follows that one can completely determine the additive structure of the Spin- and Spin^c-cobordism groups from the Anderson–Brown–Peterson splitting. We will prove an analogous 2-local splitting for MSpin^h in Section 7. In order to determine the additive structure of $\pi_*\mathrm{MSpin}^h$, we need to show that Spin^h-cobordism groups are odd torsion-free.

Proposition 2.8. *Let p be an odd prime. Then $\pi_*\mathrm{MSpin}^h$ is finitely generated in each degree and has no p-torsion.*

Proof. By Lemma 2.7, it suffices to show that

$$\pi_*(\Sigma^{-3}\mathrm{MSpin} \wedge \mathrm{MSO}(3)) \cong \mathrm{MSpin}_*\Sigma^{-3}\mathrm{MSO}(3)$$

has no p-torsion. We will argue via the Atiyah–Hirzebruch spectral sequence.[1] In the present context, this has signature

$$E^2_{s,t} = H_s(\Sigma^{-3}\mathrm{MSO}(3); \mathrm{MSpin}_t) \Longrightarrow \mathrm{MSpin}_{s+t}\Sigma^{-3}\mathrm{MSO}(3). \tag{2.1}$$

We will show that there is no p-torsion on the E^∞ page of this spectral sequence, which will imply that $\mathrm{MSpin}_*\Sigma^{-3}\mathrm{MSO}(3)$ has no p-torsion.

(i) MSpin_* is finitely generated and has no p-torsion by [Sto68, p. 336].

(ii) Let G be a finitely generated abelian group with no p-torsion. Since MSO(3) is defined as the Thom space of the universal bundle over BSO(3), the Thom isomorphism induces an isomorphism

$$\tilde{H}_s(\Sigma^{-3}\mathrm{MSO}(3); G) \cong H_s(\mathrm{BSO}(3); G).$$

Since $H^*(\mathrm{BSO}(3); \mathbb{Z})$ has no p-torsion [BH59, §30.5], the universal coefficient theorem implies that $H_*(\mathrm{BSO}(3); G)$ has no p-torsion.

[1] We learned this argument from Proposition 3.1 of Debray's lecture notes on Spin-U_2 bordism [Deb21].

(iii) The free summands of MSpin_* all lie in even degrees [Sto68, p. 340]. Similarly, the free summands of $H^*(\mathrm{BSO}(3); \mathbb{Z})$ all lie in even degrees [BH59, Proposition 30.3]. If G is a finitely generated abelian group, the universal coefficient theorem thus implies that the free summands of $H_*(\mathrm{BSO}(3); G)$ all lie in even degrees. By the Thom isomorphism, the free summands of $H_*(\Sigma^{-3}\mathrm{MSO}(3); G)$ likewise lie in even degrees.

Steps (i) and (ii) imply that there is no p-torsion on the E^2 page of Equation 2.1. Any p-torsion on the E^∞ page must therefore arise from a differential between free summands. Steps (i) and (iii) imply that no so such differentials exist, since either the source or target of any differential lies in odd degree.

Also, there are only finitely many nonzero groups on the E^∞ page for a given total degree, and each group is finitely generated, so $\pi_*\mathrm{MSpin}^h$ is finitely generated. □

2.4. **Atiyah–Bott–Shapiro map.** A critical aspect of Atiyah–Bott–Shapiro's work [ABS64] on spin geometry are the Atiyah–Bott–Shapiro orientations

$$\varphi^r : \mathrm{MSpin} \to \mathrm{KO},$$
$$\varphi^c : \mathrm{MSpin}^c \to \mathrm{KU}.$$

In analogy with φ^r and φ^c, one might hope for an Atiyah–Bott–Shapiro orientation

$$\varphi^h : \mathrm{MSpin}^h \to \mathrm{KSp}.$$

However, the lack of quaternionic tensor product prevents MSpin^h and KSp from being ring spectra, so a map $\mathrm{MSpin}^h \to \mathrm{KSp}$ cannot be an orientation. Nevertheless, Hu [Hu22, §1.3] and Freed–Hopkins [FH21, §9.2.2] independently constructed an Atiyah–Bott–Shapiro *map* φ^h that is a module map over the real Atiyah–Bott–Shapiro orientation φ^r.

Proposition 2.9 (Freed–Hopkins, Hu)**.** *There is a* quaternionic *Atiyah–Bott–Shapiro map* $\varphi^h : \mathrm{MSpin}^h \to \mathrm{KSp}$ *that is a module map over* $\varphi^r : \mathrm{MSpin} \to \mathrm{KO}$.

Proof. See [Hu22, p. 36]. □

Remark 2.10. The spectrum maps φ^r, φ^c, and φ^h are sometimes denoted in the literature by $\hat{\mathcal{A}}$, $\hat{\mathcal{A}}^c$, and $\hat{\mathcal{A}}^h$, since the real Atiyah–Bott–Shapiro orientation is the spectrum-level lift of the $\hat{\mathcal{A}}$-genus.

Traditionally, the Atiyah–Bott–Shapiro orientations (or map in the quaternionic case) are constructed in terms of Clifford algebras. Joachim gave a purely homotopical construction of the real and complex ABS orientations [Joa04], which implies that the maps φ^r and φ^c are E_∞-ring maps. It would be interesting to give an analogous construction for φ^h.

Problem 2.11. Give a homotopical construction of $\varphi^h : \mathrm{MSpin}^h \to \mathrm{KSp}$, and prove that φ^h is an E_∞-module map over the E_∞-ring map $\varphi^r : \mathrm{MSpin} \to \mathrm{KO}$.

3. Summary of the Anderson–Brown–Peterson splitting

Anderson, Brown, and Peterson's 2-local splittings of MSpin and MSpin^c [ABP67] involve extensive calculations, many of which are omitted from their write-up. In this section, we will attempt to summarize the proof strategy of Anderson–Brown–Peterson. Our proof of Theorem 1.1 is largely inspired by the strategy outlined here, as we will discuss in Section 3.2.

Notation 3.1. For $n \in \mathbb{N}$, let $\mathcal{P}(n)$ denote the set of all partitions of n, and let $\mathcal{P}_1(n)$ denote the set of all partitions of n that do not have 1 as a summand. Let $\mathcal{P} := \bigcup_{n=0}^{\infty} \mathcal{P}(n)$ be the set of all partitions, and let $\mathcal{P}_1$ be the set of all partitions that do not have 1 as a summand. Let $\mathcal{P}_{\text{even}} := \bigcup_{n=0}^{\infty} \mathcal{P}(2n)$ be the set of all even partitions, and let $\mathcal{P}_{\text{odd}} := \bigcup_{n=0}^{\infty} \mathcal{P}(2n+1)$ be the set of all odd partitions.

If $I = (i_1, \ldots, i_r)$ is a partition, we let $|I| = i_1 + \ldots + i_r$ denote the sum of I.

Notation 3.2. Unless otherwise specified, whenever we write H^* in this article, we mean cohomology with $\mathbb{Z}/2\mathbb{Z}$-coefficients. Given a ring R, we write HR to denote the associated Eilenberg–Mac Lane spectrum.

Definition 3.3. Given any spectrum X and any integer n, there is a spectrum $X\langle n\rangle$ with $\pi_k X\langle n\rangle \cong 0$ for $k < n$ and a map $X\langle n\rangle \to X$ that induces an isomorphism $\pi_k X\langle n\rangle \cong \pi_k X$ for $k \geq n$. The spectrum $X\langle n\rangle$ equipped with the map $X\langle n\rangle \to X$ is called the *n-connective cover* of X, and is unique up to unique isomorphism in the stable homotopy category.

Example 3.4. The spectra ko, ku, and ksp are the 0-connective covers (or just *connective covers*) $\mathrm{KO}\langle 0\rangle$, $\mathrm{KU}\langle 0\rangle$, and $\mathrm{KSp}\langle 0\rangle$, respectively.

We can now state the Anderson–Brown–Peterson splitting of MSpin.

Theorem 3.5 (Anderson–Brown–Peterson)**.** *There is a collection of (homogeneous) cohomology classes $Z \subset H^*\mathrm{MSpin}$ and a map of spectra*

$$\mathrm{MSpin} \to \bigvee_{k=0}^{\infty} \left(\bigvee_{\mathcal{P}_1(2k)} \mathrm{ko}\langle 8k\rangle \vee \bigvee_{\mathcal{P}_1(2k+1)} \mathrm{ko}\langle 8k+2\rangle \right) \vee \bigvee_{z \in Z} \Sigma^{\deg z} H\mathbb{Z}/2\mathbb{Z}$$

that is a 2-local homotopy equivalence.

Similarly, there is a splitting for MSpin^c:

Theorem 3.6 (Anderson–Brown–Peterson)**.** *There is a set of (homogeneous) cohomology classes $Z \subset H^*\mathrm{MSpin}^c$ and a map of spectra*

$$\mathrm{MSpin}^c \to \bigvee_{I \in \mathcal{P}} \mathrm{ku}\langle 4|I|\rangle \vee \bigvee_{z \in Z} \Sigma^{\deg z} H\mathbb{Z}/2\mathbb{Z} \tag{3.1}$$

that is a 2-local homotopy equivalence.

The proof strategy for these theorems boils down to the following four steps.

(i) Use characteristic classes to construct the maps of spectra

$$\begin{aligned}\Phi &: \mathrm{MSpin} \to \bigvee_k \big(\bigvee_{\mathcal{P}_1(2k)} \mathrm{ko}\langle 8k\rangle \vee \bigvee_{\mathcal{P}_1(2k+1)} \mathrm{ko}\langle 8k+2\rangle\big),\\ \Phi^c &: \mathrm{MSpin}^c \to \bigvee_{\mathcal{P}} \mathrm{ku}\langle 4|I|\rangle.\end{aligned}$$

The maps $\mathrm{MSpin} \to \mathrm{ko}\langle d\rangle$ come from KO-*Pontryagin classes*, whose definition we will recall in a moment. The maps $\mathrm{MSpin}^c \to \mathrm{ku}\langle d\rangle$ are not explicitly discussed in [ABP67], but these come from KU-characteristic classes. Both KO- and KU-characteristic classes are indexed by integer partitions, which accounts for the role of partitions in the 2-local splitting theorems.

(ii) Assuming that there are maps

$$\begin{aligned}\Psi &: \mathrm{MSpin} \to \bigvee_k \big(\bigvee_{\mathcal{P}_1(2k)} \mathrm{ko}\langle 8k\rangle \vee \bigvee_{\mathcal{P}_1(2k+1)} \mathrm{ko}\langle 8k+2\rangle\big) \vee \bigvee_z \Sigma^{\deg z} H\mathbb{Z}/2\mathbb{Z},\\ \Psi^c &: \mathrm{MSpin}^c \to \bigvee_{\mathcal{P}} \mathrm{ku}\langle 4|I|\rangle \vee \bigvee_z \Sigma^{\deg z} H\mathbb{Z}/2\mathbb{Z}.\end{aligned}$$

inducing isomorphisms on mod 2-cohomology, deduce that Ψ and Ψ^c are 2-local equivalences.

A map of spectra $X \to Y$ that induces an isomorphism on mod 2 cohomology is a 2-complete equivalence. If the homotopy groups of X and Y are all finitely generated (as is the case for all spectra that we will consider), then a 2-complete equivalence is a 2-local equivalence.

(iii) Prove that Φ and Φ^c induce isomorphisms on certain Margolis homologies.

In general, Margolis homology is easier to compute than mod 2 cohomology. Knowing that Φ and Φ^c induce isomorphisms on Margolis homology acts as the base case of an induction argument to prove that Φ and Φ^c induce isomorphisms on mod 2 cohomology.

(iv) By identifying a suitable collection of ordinary cohomology classes of MSpin and MSpin^h, form the maps Ψ and Ψ^c and prove that these induce isomorphisms on mod 2 cohomology.

Surjectivity is the easier part of this step. For injectivity, filter the source and target cohomologies by degree and show that if Ψ and Ψ^c induce isomorphisms on cohomology in degrees at most n, then Ψ and Ψ^c are injective on cohomology in degrees at most $n+1$.

3.1. KO**-Pontryagin classes.** The key to splitting MSpin and MSpin^c are KO-Pontryagin classes, since these give us maps from MSpin and MSpin^c to the various K-theoretic summands in the splitting. These were first introduced in [ABP66, §4], but we will recall the definition here.

Definition 3.7. The i^{th} KO-*Pontryagin class* of an oriented vector bundle V on X is the unique class $\pi^i(V) \in \mathrm{KO}^0(X)$ such that

(i) π^i is natural in V for all i;

(ii) for each complex line bundle L, we have

- $\pi^0(L) = 0$,
- $\pi^1(L) = L - 2$, and
- $\pi^i(L) = 0$ for $i \geq 2$;

(iii) for any oriented bundles V and W, we have

$$\sum_{i\geq 0} \pi^i(V \oplus W)t^i = \left(\sum_{j\geq 0} \pi^j(V)t^j\right)\left(\sum_{k\geq 0} \pi^k(W)t^k\right).$$

Given a partition $I = (i_1, \ldots, i_n)$, the I^{th} KO-Pontryagin class is the product $\pi^I := \pi^{i_1} \cdots \pi^{i_n}$.

The fact these three properties characterize π^i (and hence π^I) follows from [ABP66, Proposition 4.4]. The classes $\pi^I \in \mathrm{KO}^0(\mathrm{BSpin})$ determine maps $\mathrm{MSpin} \to \mathrm{KO}\langle d\rangle$ by multiplication with $\varphi^r : \mathrm{MSpin} \to \mathrm{KO}$, where the degree d of connectivity is determined by the degree of π^I (which are given in [ABP67, Theorem 2.1]).

3.2. **Proof strategy for splitting** MSpin^h**.** Here is our strategy for proving Theorem 1.1:

(i) Compute the homotopy groups and cohomology of the spectrum F. Then use KO-Pontryagin classes to build maps $\mathrm{MSpin}^h \to \mathrm{ksp}\langle 4|I|\rangle$ for each partition I. For odd partitions I, show that these maps lift to maps $\mathrm{MSpin}^h \to \Sigma^{4|I|}F$. Using some spectral sequence and characteristic class computations, describe what each of these maps does in cohomology (after choosing the correct lifts with some obstruction theory). Then, take wedge sums to form the map

$$\Phi^h : \mathrm{MSpin}^h \to \bigvee_{k=0}^{\infty}\left(\bigvee_{\mathcal{P}(2k)} \mathrm{ksp}\langle 8k\rangle \vee \bigvee_{\mathcal{P}(2k+1)} \Sigma^{8k+4}F\right).$$

(ii) Prove that Φ^h induces isomorphisms on Margolis homology by computing the Margolis homology of the cohomology of each summand and of $H^*\mathrm{MSpin}^h$. As in the MSpin and MSpin^c cases, this is almost everything we need to get an isomorphism in cohomology.

(iii) Find a set of cohomology classes $Z \subset H^*\mathrm{MSpin}^h$ such that the map

$$\mathrm{MSpin}^h \to \bigvee_{k=0}^{\infty}\left(\bigvee_{\mathcal{P}(2k)} \mathrm{ksp}\langle 8k\rangle \vee \bigvee_{\mathcal{P}(2k+1)} \Sigma^{8k+4}F\right) \vee \bigvee_{z\in Z} \Sigma^{\deg z} H\mathbb{Z}/2\mathbb{Z}$$

induces a surjection in cohomology. Then, using the fact that Φ^h gives isomorphisms on Margolis homology, filter the Steenrod modules on both sides of this map by the lowest degree in which summands are nonzero and use this to show that Φ^h induces an injection as well, giving us an isomorphism in mod 2 cohomology.

(iv) The isomorphism on mod 2 cohomology gives an equivalence of spectra in the 2-complete category, and this is a 2-local equivalence due to finitely generated homotopy groups.

The overall plan is analogous to the strategy used in [ABP67]. In steps (ii) and (iii), we have to make a few adjustments to deal with the fact that MSpin^h is not a ring spectrum, but instead a module spectrum over MSpin.

4. Cohomology of BSpin^h and MSpin^h

In Section 5, we will construct characteristic classes that realize the non-Eilenberg–Mac Lane summands of our splitting map. To do this, we need a few cohomological computations, which we collect in this section.

First, we present the cohomology of the classifying spaces BSpin, BSpin^c, and BSpin^h.

Proposition 4.1. *The cohomology of* BSpin *is the ring*
$$H^*\mathrm{BSpin} \cong \mathbb{Z}/2\mathbb{Z}\left[w_i \;\middle|\; i \geq 2, i \neq 2^k+1 \text{ for } k \geq 0\right],$$
where w_i is the i^{th} Stiefel–Whitney class of the canonical oriented bundle $\mathrm{BSpin} \to \mathrm{BSO}$.

Proof. See [Sto68, p. 292]. □

Proposition 4.2. *The cohomology of* BSpin^c *is the ring*
$$H^*\mathrm{BSpin}^c \cong \mathbb{Z}/2\mathbb{Z}\left[w_i \;\middle|\; i \geq 2, i \neq 2^{k+1}+1 \text{ for } k \geq 0\right],$$
where w_i is the i^{th} Stiefel–Whitney class of the canonical oriented bundle $\mathrm{BSpin}^c \to \mathrm{BSO}$.

Proof. See [Sto68, p. 293]. □

Proposition 4.3. *The cohomology of* BSpin^h *is the ring*
$$H^*\mathrm{BSpin}^h \cong \mathbb{Z}/2\mathbb{Z}\left[w_i \;\middle|\; i \geq 2, i \neq 2^{k+2}+1 \text{ for } k \geq 0\right],$$
where w_i is the i^{th} Stiefel–Whitney class of the canonical oriented bundle $\mathrm{BSpin}^h \to \mathrm{BSO}$. *The Stiefel–Whitney class w_5 vanishes.*

Proof. See [Hu22, Proposition 2.31]. □

Remark 4.4. Note that the classes w_{2^k+1} do not vanish in general, but are non-zero polynomials in lower Stiefel–Whitney classes. For BSpin, one can find these relations by noting that $w_2 = 0$ for degree reasons, imposing the relation $\mathrm{Sq}^{2^{k-1}} \cdots \mathrm{Sq}^2\mathrm{Sq}^1 w_2 = 0$, and applying the Wu formula. For BSpin^h, the classes w_{2^k+1} are determined by $\mathrm{Sq}^{2^{k-1}} \cdots \mathrm{Sq}^4 w_5 = 0$.

By pulling back the KO-Pontryagin class π^I under $\mathrm{BSpin}^h \to \mathrm{BSO}$, we get a KO-Pontryagin class π_h^I for BSpin^h. A fact we will need later is that the associated map $\mathrm{BSpin}^h \to \mathrm{KO}$ admits a lift to $\mathrm{ko}\langle 4|I|\rangle$ if $I \in \mathcal{P}_{\text{even}}$ or to $\mathrm{ko}\langle 4|I|-2\rangle$ if $I \in \mathcal{P}_{\text{odd}}$.

Proposition 4.5. *The map* $\mathrm{BSpin}^h \to \mathrm{BSO} \xrightarrow{\pi_h^I} \mathrm{KO}$ *admits a lift to* $\mathrm{ko}\langle 4|I|\rangle$ *if* $I \in \mathcal{P}_{\mathrm{even}}$ *or to* $\mathrm{ko}\langle 4|I|-2\rangle$ *if* $I \in \mathcal{P}_{\mathrm{odd}}$.

Proof. Since all torsion in the integral cohomology is order two (see [Hu22, Corollary 2.36]), we see that the Pontryagin class $p_I = p_{i_1}\dots p_{i_r}$ corresponding to a partition I is non-torsion, since its reduction modulo two is $w_{2i_1}^2 \dots w_{2i_r}^2$ and we know this is not zero. So p_I is nonzero after rationalization. Moreover, there is no integral class x such that $2x = p_I$ after rationalization, since this would imply $p_I - 2x$ is a torsion class, which can then be written as δy for some mod 2 cohomology class y, where δ is the Bockstein homomorphism. Reducing mod 2, we see that $w_{2i_1}^2 \dots w_{2i_r}^2 = \mathrm{Sq}^1 y$. This contradicts Lemma 6.11, the proof of which we save for our discussion of Margolis homology.

Hence the hypotheses of the proposition of [Sto68, p. 303, 304] are met, so for $|I|$ even, π_R^I admits a lift to $\mathrm{ko}\langle 4|I|\rangle$ with $x_{4|I|}$ mapping to $p_I + \mathrm{Sq}^3\mathrm{Sq}^1\alpha$ for some α after reduction mod 2, and for $|I|$ odd, π_R^I admits a lift to $\mathrm{ko}\langle 4|I|-2\rangle$ such that if x is the image of $x_{4|I|-2}$, $\mathrm{Sq}^2 x = p_I$ ([Sto68, p. 314]). ☐

Propositions 4.1, 4.2, and 4.3 immediately determine the cohomology of MSpin, MSpin^c, and MSpin^h via the Thom isomorphism. The action of the Steenrod algebra on each of these modules is determined by the rule $\mathrm{Sq}^i u = w_i u$, where u is the Thom class of any bundle (see [MS74, p. 91]). The maps between the Thom spectra induce maps

$$H^*\mathrm{MSpin}^h \longrightarrow H^*\mathrm{MSpin}^c \longrightarrow H^*\mathrm{MSpin}$$

of cohomology, with Thom classes mapping to Thom classes. Since the maps of classifying spaces are maps over BSO, Stiefel–Whitney classes map to the corresponding Stiefel–Whitney classes.

4.1. **Steenrod modules.** Modules over the Steenrod algebra are ubiquitous in [ABP67], as well as the present paper. Indeed, if a map of spectra $X \to Y$ is to be a 2-local equivalence, then one needs to show that the induced map $H^*Y \to H^*X$ is an isomorphism of modules over the mod 2 Steenrod algebra. In this section, we collect a few results about the cohomology of various connective covers of ko, ku, and ksp in terms of Steenrod modules.

Notation 4.6. Throughout this article, $\mathcal{A}$ will denote the mod 2 Steenrod algebra.

Proposition 4.7. *Suppose* $k = 0, 1, 2, 4 \pmod 8$*. Then there is a class* $x_k \in H^k\mathrm{ko}\langle k\rangle$ *such that the map*

$$\begin{aligned} \mathcal{A} &\to H^*\mathrm{ko}\langle k\rangle \\ 1 &\mapsto x_k \end{aligned}$$

induces an isomorphism $\mathcal{A}/I_k \to H^*\mathrm{ko}\langle k\rangle$, where $I_k \subset \mathcal{A}$ is the left ideal

$$I_k = \begin{cases} \mathcal{A}\mathrm{Sq}^1 + \mathcal{A}\mathrm{Sq}^2 & k = 0 \pmod 8, \\ \mathcal{A}\mathrm{Sq}^2 & k = 1 \pmod 8, \\ \mathcal{A}\mathrm{Sq}^3 & k = 2 \pmod 8, \\ \mathcal{A}\mathrm{Sq}^1 + \mathcal{A}\mathrm{Sq}^5 & k = 4 \pmod 8. \end{cases}$$

Proof. See [Sto68, p. 295]. □

Using the Bott periodicity isomorphism $\mathrm{KSp} \cong \Sigma^4\mathrm{KO}$ and the uniqueness of connective covers, we see that $\Sigma^4\mathrm{ko}\langle k\rangle \cong \mathrm{ksp}\langle k+4\rangle$, giving us the following result in cohomology:

Corollary 4.8. *If $k = 0, 4, 5, 6 \pmod 8$, there is a class $y_k \in H^k\mathrm{ksp}\langle k\rangle$ such that the map*

$$\begin{aligned} \mathcal{A} &\to H^*\mathrm{ksp}\langle k\rangle \\ 1 &\mapsto y_k \end{aligned}$$

induces an isomorphism $\mathcal{A}/I_k \to H^*\mathrm{ksp}\langle k\rangle$, where $I_k \subset \mathcal{A}$ is the left ideal

$$I_k = \begin{cases} \mathcal{A}\mathrm{Sq}^1 + \mathcal{A}\mathrm{Sq}^5 & k = 0 \pmod 8, \\ \mathcal{A}\mathrm{Sq}^1 + \mathcal{A}\mathrm{Sq}^2 & k = 4 \pmod 8, \\ \mathcal{A}\mathrm{Sq}^2 & k = 5 \pmod 8, \\ \mathcal{A}\mathrm{Sq}^3 & k = 6 \pmod 8. \end{cases}$$

Proof. By Bott periodicity, this is a degree 4 shift of Proposition 4.7. □

We also describe the cohomology of ku and its role in the splitting of MSpin^c, as it will be relevant later.

Proposition 4.9. *For each k, there is a class $z_{2k} \in H^{2k}\mathrm{ku}\langle 2k\rangle$ such that the map*

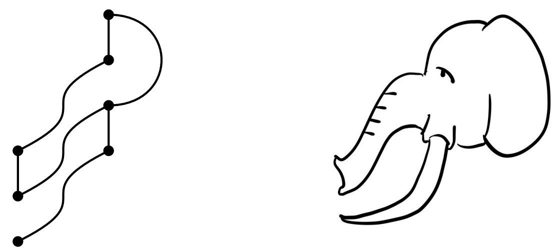
$$\begin{aligned} \mathcal{A} &\to H^*\mathrm{ku}\langle 2k\rangle \\ 1 &\mapsto z_{2k} \end{aligned}$$

induces an isomorphism $\mathcal{A}/(\mathcal{A}\mathrm{Sq}^1 + \mathcal{A}\mathrm{Sq}^3) \to H^*\mathrm{ku}\langle 2k\rangle$.

Proof. See [Sto68, p. 295]. □

The image of z_{4k} under the map $H^*\mathrm{ku}\langle 4k\rangle \to H^*\mathrm{MSpin}^c$ is particularly tractable.

Lemma 4.10. *The splitting in Equation 3.1 If $I \in \mathcal{P}$ is a partition, then the map $\mathrm{MSpin}^c \to \mathrm{ku}\langle 4|I|\rangle$ in Equation 3.1 induces $z_{4|I|} \mapsto p_I U_c$ in cohomology, where $U_c \in H^*\mathrm{MSpin}^c$ is the Thom class and p_I is the I^{th} Pontryagin class.*



(A) The elephant E (B) An elephant, unnamed

FIGURE 1. The $\mathcal{A}_1$-module E and its namesake

Proof. It is shown that the complexification of the KO-Pontryagin classes can be chosen so that $z_{4|I|} \mapsto p_I$ in [Sto68, p. 304]. The fact that multiplying with the orientation $\mathrm{MSpin}^c \to \mathrm{KU}$ induces $z_{4|I|} \mapsto p_I U_c$ is shown in [Sto68, p. 317]. The splitting map with this property is assembled at [Sto68, p. 319]. □

We now define three Steenrod modules that will show up when we compute the cohomology of various spectra. Proposition 4.7 states that $H^*\mathrm{ko}$ is a module over the subalgebra in $\mathcal{A}$ generated by Sq^0, Sq^1, and Sq^2.

Definition 4.11. Let $\mathcal{A}_1$ denote the subalgebra of $\mathcal{A}$ generated by $\mathrm{Sq}^0, \mathrm{Sq}^1$, and Sq^2. Note that $\mathcal{A}_1$ is often denoted $\mathcal{A}(1)$ in the literature. Our choice of notation is both an homage to the notation used in [ABP67] and an effort to declutter many equations in the sequel.

The following $\mathcal{A}_1$-module occurs as a summand of $H^*\mathrm{MSpin}^h$ (as will be discussed in Section 5).

Definition 4.12. The *elephant* E is the $\mathcal{A}_1$-submodule of $\Sigma^{-1}\mathcal{A}_1$ generated by Sq^1 and Sq^2 (see Figure 1).[2]

The next $\mathcal{A}_1$-module is well-known.

Definition 4.13. The *upside-down question mark* is the $\mathcal{A}_1$-module

$$\text{¿} := \mathcal{A}_1/(\mathcal{A}_1\mathrm{Sq}^1 + \mathcal{A}_1(\mathrm{Sq}^5 + \mathrm{Sq}^4\mathrm{Sq}^1))$$

(see Figure 2A).

Finally, we also define a module that generates the summands of $H^*\mathrm{MSpin}^c$.

Definition 4.14. Let C denote the $\mathcal{A}_1$-module $\mathcal{A}_1/(\mathcal{A}_1\mathrm{Sq}^1 + \mathcal{A}_1\mathrm{Sq}^3)$ (see Figure 2B).

[2]The elephant appears in [BC18] under the name R_2.

(A) The module ¿ (B) The module C

FIGURE 2. The $\mathcal{A}_1$-modules ¿ and C

5. KSp-Pontryagin and elephant classes

We now begin constructing the map given in Theorem 1.1. In this section, we will give the maps to the summands $\mathrm{ksp}\langle 8n\rangle$ and $\Sigma^{8n+4}F$ (see Definition 5.2) by defining characteristic classes for the cohomology theories defined by these spectra.

Since KSp is a KO-module, there is a map $\mathrm{KO}\wedge\mathrm{KSp}\to\mathrm{KSp}$ satisfying the usual axioms in the homotopy category. The smash product $\mathrm{ko}\langle n\rangle\wedge\mathrm{ksp}\langle m\rangle$ is $(n+m-1)$-connected, so there is a unique map $\mathrm{ko}\langle n\rangle\wedge\mathrm{ksp}\langle m\rangle\to\mathrm{ksp}\langle n+m\rangle$ fitting into the diagram

$$\begin{tikzcd} \mathrm{ko}\langle n\rangle\wedge\mathrm{ksp}\langle m\rangle \arrow[r, dashed] \arrow[d] & \mathrm{ksp}\langle n+m\rangle \arrow[d] \\ \mathrm{KO}\wedge\mathrm{KSp} \arrow[r] & \mathrm{KSp}. \end{tikzcd} \tag{5.1}$$

Recall that there is a map $\varphi^h:\mathrm{MSpin}^h\to\mathrm{ksp}$ such that $y_0\mapsto U_h$ in cohomology [Hu22, Remark 3.26], where $y_0\in H^*\mathrm{ksp}$ is the class mentioned in Corollary 4.8 and $U_h\in H^*\mathrm{MSpin}^h$ is the Thom class. For any partition I, the KO-Pontryagin class $\pi_h^I\in\mathrm{ko}\langle n\rangle^0\mathrm{BSpin}^h$ determines a class on MSpin^h through the composite

$$\mathrm{MSpin}^h\longrightarrow\mathrm{BSpin}^h\wedge\mathrm{MSpin}^h\xrightarrow{\pi_h^I\wedge\varphi^h}\mathrm{ko}\langle n\rangle\wedge\mathrm{ksp}\longrightarrow\mathrm{ksp}\langle n\rangle. \tag{5.2}$$

Here, the map $\mathrm{MSpin}^h\to\mathrm{BSpin}^h\wedge\mathrm{MSpin}^h$ is the Thom diagonal. While φ^h is not a Thom class in the sense of an orientation with respect to a ring spectrum, the principle of transferring classes from a base space to the Thom spectrum via multiplication is the same. By looking for copies of $H^*\mathrm{ksp}\langle n\rangle$ in $H^*\mathrm{MSpin}^h$, we get a sense of what classes $\pi_h^I\in\mathrm{ko}\langle n\rangle^0\mathrm{BSpin}^h$ we need. This is the method we will use to generate all of the maps that decompose MSpin^h, besides those to Eilenberg–Mac Lane spectra, which originate in ordinary cohomology.

5.1. **Module structure for** $\mathrm{ksp}\langle n\rangle$**.** The crux of understanding Equation 5.2 is the behavior of the maps $\mathrm{ko}\langle n\rangle\wedge\mathrm{ksp}\to\mathrm{ksp}\langle n\rangle$ in cohomology. It turns out that we will only need the cases $n=8k$ and $n=8k+2$ in order to prove Theorem 1.1.

For the case $n=8k$, we can use the KO-module structure of KSp.

Lemma 5.1. *The map* $\mathrm{ko}\langle 8k\rangle\wedge\mathrm{ksp}\to\mathrm{ksp}\langle 8k\rangle$ *induces* $y_{8k}\mapsto x_{8k}\otimes y_0$ *in cohomology.*

Proof. Since ko is a ring spectrum, its cohomology $A = H^*\mathrm{ko}$ can be equipped with the structure of a coalgebra. In particular, the diagram

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\mathrm{id}_A"] & A \otimes A \arrow[d, "\epsilon\otimes\mathrm{id}_A"] \\ A & \mathbb{Z}/2\mathbb{Z} \otimes A \arrow[l] \end{tikzcd}$$

commutes, where $\Delta : A \to A \otimes A$ and $\epsilon : A \to \mathbb{Z}/2\mathbb{Z}$ are $\mathbb{Z}/2\mathbb{Z}$-linear maps and the bottom arrow is the canonical isomorphism. Since the only element of A of degree zero is x_0, we can write $\Delta x_0 = a x_0 \otimes x_0$ for some coefficient $a \in \mathbb{Z}/2\mathbb{Z}$. The diagram above says $x_0 = a\epsilon(x_0)x_0$, so $a = \epsilon(x_0) = 1$. Next, let $B = H^*\mathrm{ksp}$. Since ksp is a ko-module spectrum, B can be given the structure of an A-comodule. In particular, the diagram

$$\begin{tikzcd} B \arrow[r, "\mu"] \arrow[d, "\mathrm{id}_B"] & A \otimes B \arrow[d, "\epsilon\otimes\mathrm{id}_B"] \\ B & \mathbb{Z}/2\mathbb{Z} \otimes B \arrow[l] \end{tikzcd}$$

commutes. Since $y_0 \in \mathrm{ksp}$ is the only element of degree zero, we have $\mu y_0 = b x_0 \otimes y_0$ for some $b \in \mathbb{Z}/2\mathbb{Z}$. The diagram above then says that $y_0 = b\epsilon(x_0)y_0$, so $b = 1$, and therefore the map $\mathrm{ko} \wedge \mathrm{ksp} \to \mathrm{ksp}$ has $y_0 \mapsto x_0 \otimes y_0$ in cohomology.

Finally, using $\Sigma^{8k}\mathrm{ko} \cong \mathrm{ko}\langle 8k\rangle$ and $\Sigma^{8k}\mathrm{ksp} \cong \mathrm{ksp}\langle 8k\rangle$, taking suspensions of the map $\mathrm{ko}\wedge\mathrm{ksp} \to \mathrm{ksp}$ gives us a map $\mathrm{ko}\langle 8k\rangle\wedge\mathrm{ksp} \to \mathrm{ksp}\langle 8k\rangle$ with $y_{8k} \mapsto x_{8k}\otimes y_0$ in cohomology, since $\Sigma^{8k}(\mathrm{ko}\wedge\mathrm{ksp}) \cong \Sigma^{8k}\mathrm{ko}\wedge\mathrm{ksp}$. We just have to check that this is the original map we were concerned with. Recall that the desired map $\mathrm{ko}\langle 8k\rangle \wedge \mathrm{ksp} \to \mathrm{ksp}\langle 8k\rangle$ is the unique map making Diagram 5.1 commute. It thus suffices to show that the diagram

$$\begin{tikzcd} \Sigma^{8k}\mathrm{ko} \wedge \mathrm{ksp} \arrow[r] \arrow[d] & \Sigma^{8k}\mathrm{ksp} \arrow[d] \\ \mathrm{KO} \wedge \mathrm{KSp} \arrow[r] & \mathrm{KSp} \end{tikzcd} \tag{5.3}$$

commutes. When $k = 0$, Diagram 5.3 is a special case of Diagram 5.1 and hence commutes. Suspending $8k$ times gives us the desired diagram, except we must verify that the bottom edge is still the module multiplication map. But this is true because $\mathrm{KSp} \cong \Sigma^4\mathrm{KO}$ and $\Sigma^8\mathrm{KO} \cong \mathrm{KO}$ as KO-module spectra (Proposition 2.1). □

5.2. **The elephant spectrum.** The case $n = 8k+2$ is considerably more complicated. For $n = 8k$, one can find $H^*\mathrm{ksp}\langle n\rangle$ summands in the cohomology of MSpin^h, but it appears that the cohomology of a different spectrum arises at $n = 8k+2$. This leads us to the following definition.

Definition 5.2. Consider the map $\mathrm{ko} \to H\mathbb{Z}$ inducing an isomorphism on π_0. Composing with the quotient map $H\mathbb{Z} \to H\mathbb{Z}/2\mathbb{Z}$, we get a map $\mathrm{ko} \to H\mathbb{Z}/2\mathbb{Z}$. Define the *elephant spectrum*[3] $F := \mathrm{fib}(\mathrm{ko} \to H\mathbb{Z}/2\mathbb{Z})$ as the fiber of this map.

[3] This name will be justified in Lemma 5.4.

Shifting by $8k+4$, we observe fiber sequences

$$\Sigma^{8k+4}F \longrightarrow \mathrm{ksp}\langle 8k+4\rangle \longrightarrow \Sigma^{8k+4}H\mathbb{Z}/2\mathbb{Z}. \tag{5.4}$$

We can now readily compute the homotopy and cohomology of F.

Lemma 5.3. *For $k<0$, we have $\pi_k F \cong 0$. For $k \geq 0$, we have*

$$\pi_k F \cong \begin{cases} \mathbb{Z} & k = 0,4 \pmod 8, \\ \mathbb{Z}/2\mathbb{Z} & k = 1,2 \pmod 8, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. For all k, we have exact sequences

$$\pi_{k+1}\mathrm{ko} \longrightarrow \pi_{k+1}H\mathbb{Z}/2\mathbb{Z} \longrightarrow \pi_k F \longrightarrow \pi_k\mathrm{ko} \longrightarrow \pi_k H\mathbb{Z}/2\mathbb{Z} \tag{5.5}$$

from the long exact sequence of a fibration.

(i) If k is not 0 or -1, then $\pi_{k+1}H\mathbb{Z}/2\mathbb{Z} \cong \pi_k H\mathbb{Z}/2\mathbb{Z} \cong 0$ and thus $\pi_k F \to \pi_k\mathrm{ko}$ is an isomorphism.

(ii) If $k=0$, then $\pi_1 H\mathbb{Z}/2\mathbb{Z} \cong 0$, so $\pi_0 F$ is the kernel of the quotient map $\mathbb{Z} \cong \pi_0\mathrm{ko} \to \pi_0 H\mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$. Since $\pi_0 F$ is the kernel of the quotient map $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$, it follows that $\pi_0 F \to \pi_0\mathrm{ko}$ can be identified with the inclusion $2\mathbb{Z} \to \mathbb{Z}$.

(iii) Finally, suppose $k=-1$. The map $\pi_0\mathrm{ko} \to \pi_0 H\mathbb{Z}/2\mathbb{Z}$ is an epimorphism, so $\pi_0 H\mathbb{Z}/2\mathbb{Z} \to \pi_{-1}F$ is zero and $\pi_{-1}H\mathbb{Z}/2\mathbb{Z} \cong 0$. Exactness of Equation 5.5 implies that $\pi_{-1}F \to \pi_{-1}\mathrm{ko}$ is an isomorphism. □

Lemma 5.4. *The cohomology of F is given by $H^*F \cong \mathcal{A} \otimes_{\mathcal{A}_1} E$, where E is the elephant (see Definition 4.12).*

Proof. By the definition of E, we have a short exact sequence

$$0 \longrightarrow \Sigma^1 E \longrightarrow \mathcal{A}_1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

Since $\mathcal{A}$ is flat (in fact, free) as a right $\mathcal{A}_1$-module, tensoring gives us a short exact sequence

$$0 \longrightarrow \mathcal{A} \otimes_{\mathcal{A}_1} \Sigma^1 E \longrightarrow \mathcal{A} \xrightarrow{\phi} \mathcal{A} \otimes_{\mathcal{A}_1} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0 \tag{5.6}$$

of $\mathcal{A}$-modules. Recall that $H^*H\mathbb{Z}/2\mathbb{Z} \cong \mathcal{A}$ and $H^*\mathrm{ko} \cong \mathcal{A} \otimes_{\mathcal{A}_1} \mathbb{Z}/2\mathbb{Z}$. Since the map $\mathrm{ko} \to H\mathbb{Z}/2\mathbb{Z}$ is non-trivial, it must represent the bottom class of $H^*\mathrm{ko}$ and therefore induces the map $\phi : \mathcal{A} \to \mathcal{A} \otimes_{\mathcal{A}_1} \mathbb{Z}/2\mathbb{Z}$ in Equation 5.6.

The fiber sequence defining F gives us a long exact sequence

$$H^*H\mathbb{Z}/2\mathbb{Z} \longrightarrow H^*\mathrm{ko} \longrightarrow H^*F \longrightarrow H^{*+1}H\mathbb{Z}/2\mathbb{Z} \longrightarrow H^{*+1}\mathrm{ko} \tag{5.7}$$

in cohomology. Since $H^*H\mathbb{Z}/2\mathbb{Z} \to H^*\mathrm{ko}$ is an epimorphism, Equation 5.7 induces exact sequences

$$0 \longrightarrow H^*F \longrightarrow H^{*+1}H\mathbb{Z}/2\mathbb{Z} \longrightarrow H^{*+1}\mathrm{ko}.$$

Thus H^*F is the kernel of the map $H^*H\mathbb{Z}/2\mathbb{Z} \to H^*\mathrm{ko}$ shifted by -1. That is, $H^*F \cong \Sigma^{-1}\ker\phi$, so Equation 5.6 implies $H^*F \cong \mathcal{A} \otimes_{\mathcal{A}_1} E$. □

Remark 5.5. Note that while the homotopy groups of F are abstractly isomorphic to those of ko, they have a different structure as a module over $\pi_*\mathbb{S}$. This can be seen in the Adams spectral sequence for $\mathcal{A} \otimes_{\mathcal{A}_1} E$ in [BC18, Figure 29] (note that E is referred to as R_2 in *loc. cit.*).

5.3. **Aside on integral cohomology.** The cohomology of $\Sigma^{8k+4}F$ arises in the cohomology of MSpin^h, which leads us to look for *elephant classes* $\mathrm{MSpin}^h \to \Sigma^{8k+4}F$. We will build these using KO-Pontryagin classes once we know how to lift $\mathrm{ko}\langle 8k+2\rangle \wedge \mathrm{ksp} \to \mathrm{ksp}\langle 8k+2\rangle$ to $\mathrm{ko}\langle 8k+2\rangle \wedge \mathrm{ksp} \to \Sigma^{8k+4}F$. To do this, we need a few results in integral cohomology.

Lemma 5.6. *Let* $\mathrm{pr}_i : \mathrm{BU} \times \mathrm{BU} \to \mathrm{BU}$ *denote projection onto the* i^{th} *factor for* $i = 1, 2$. *Let* $\gamma \to \mathrm{BU}$ *be the classifying virtual bundle. Let* $\alpha := \mathrm{pr}_1^*\gamma$ *and* $\beta := \mathrm{pr}_2^*\gamma$, *so that the external tensor product* $\alpha \otimes \beta$ *is a virtual bundle on* $\mathrm{BU} \times \mathrm{BU}$. *Then in* $H^*(\mathrm{BU} \times \mathrm{BU}; \mathbb{Z})$, *we have*

$$c_4(\alpha \otimes \beta) = -6c_2(\alpha)c_2(\beta) \pmod{c_1(\alpha), c_1(\beta)}. \tag{5.8}$$

Proof. This can be computed using the Chern character. Recall that the Chern character of a virtual bundle ξ with rank n is defined to be

$$\mathrm{ch}(\xi) = n + \sum_{k=1}^{\infty} \frac{s_k(c(\xi))}{k!},$$

where the s_k are polynomials of (cohomological) degree k in the Chern classes [MS74, pp. 188]. In particular, the first four s_k are

$$\begin{aligned}
s_1(c(\xi)) &= c_1(\xi),\\
s_2(c(\xi)) &= c_1(\xi)^2 - 2c_2(\xi),\\
s_3(c(\xi)) &= c_1(\xi)^3 - 3c_1(\xi)c_2(\xi) + 3c_3(\xi),\\
s_4(c(\xi)) &= c_1(\xi)^4 - 4c_1(\xi)^2c_2(\xi) + 2c_2(\xi)^2 + 4c_1(\xi)c_3(\xi) - 4c_4(\xi).
\end{aligned}$$

Working modulo the ideal generated by $c_1(\alpha)$ and $c_1(\beta)$, we have

$$\mathrm{ch}(\alpha) = -c_2(\alpha) + \frac{1}{2}c_3(\alpha) + \cdots \pmod{c_1(\alpha), c_1(\beta)},$$

and similarly for $\mathrm{ch}(\beta)$. Since the Chern character is multiplicative over tensor products, we see that

$$\mathrm{ch}(\alpha \otimes \beta) = c_2(\alpha)c_2(\beta) + \text{higher degree terms} \pmod{c_1(\alpha), c_1(\beta)}.$$

Thus $c_1(\alpha\otimes\beta) = c_2(\alpha\otimes\beta) = c_3(\alpha\otimes\beta) = 0 \pmod{c_1(\alpha), c_1(\beta)}$, and $c_4(\alpha\otimes\beta)$ is given by the equation

$$-\frac{4c_4(\alpha\otimes\beta)}{4!} = c_2(\alpha)c_2(\beta) \pmod{c_1(\alpha), c_1(\beta)}.$$

Solving for $c_4(\alpha\otimes\beta)$ gives the desired result. $\square$

Next, we compute $H^*(K(\mathbb{Z},3);\mathbb{Z})$ in a small range. We will do this with the Serre spectral sequence (see [Hat04, Example 5.20]), but one can alternatively apply the universal coefficient theorem to Cartan's computation of $H_*(K(G,n);\mathbb{Z})$ [Car55].

Lemma 5.7. *In degrees at most 8, the integral cohomology of $K(\mathbb{Z},3)$ is*

$$H^i(K(\mathbb{Z},3);\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i = 0,3, \\ 0 & i = 1,2,3,4,5,7, \\ \mathbb{Z}/2\mathbb{Z} & i = 6, \\ \mathbb{Z}/3\mathbb{Z} & i = 8. \end{cases}$$

Proof. We know that $H^0(K(\mathbb{Z},3);\mathbb{Z}) \cong \mathbb{Z}$ and $H^1(K(\mathbb{Z},3);\mathbb{Z}) \cong H^2(K(\mathbb{Z},3);\mathbb{Z}) \cong 0$ by the Hurewicz theorem and universal coefficient theorem. The path-loop fibration gives us a fiber sequence

$$\mathbb{CP}^\infty \cong K(\mathbb{Z},2) \longrightarrow * \longrightarrow K(\mathbb{Z},3).$$

The Serre spectral sequence associated with this fibration has signature

$$E_2^{p,q} = H^p(K(\mathbb{Z},3); H^q(\mathbb{CP}^\infty;\mathbb{Z})) \Longrightarrow H^{p+q}(*;\mathbb{Z}).$$

Since $H^*(\mathbb{CP}^\infty;\mathbb{Z})$ is a polynomial ring generated by an element $x \in H^2(\mathbb{CP}^\infty;\mathbb{Z})$, the row $E_2^{*,2k}$ is given by $H^*(K(\mathbb{Z},3);\mathbb{Z})$ times x^k, and the odd rows $E_2^{*,2k+1}$ vanish (see Figure 3).

(i) Since the spectral sequence converges to the cohomology of a contractible space, the class $x \in E_2^{0,2}$ must be nonzero under some differential. Moreover, $H^1(K(\mathbb{Z},3);\mathbb{Z}) \cong H^2(K(\mathbb{Z},3);\mathbb{Z}) \cong 0$ implies that $\ker d_3 \cong \operatorname{coker} d_3 \cong 0$, so $d_3 : E_3^{0,2} \to E_3^{3,0}$ is an isomorphism. In particular, $H^3(K(\mathbb{Z},3);\mathbb{Z}) \cong \mathbb{Z}$. Denote $\iota := d_3(x)$, which generates $H^3(K(\mathbb{Z},3);\mathbb{Z})$.

(ii) Note that $E_2^{4,0} \cong H^4(K(\mathbb{Z},3);\mathbb{Z})$ must be 0, because any differential hitting $E_r^{4,0}$ has domain 0 and $E^{4,0}$ converges to $H^4(*;\mathbb{Z}) = 0$.

(iii) Similarly, the only possible non-zero differential hitting $E_r^{5,0}$ is $d_5 : E_5^{0,4} \to E_5^{5,0}$. But $E_2^{0,4}$ is generated by x^2, and $d_3(x^2) = 2xd_3(x) = 2x\iota$ is non-zero and non-torsion. Thus $E_{\geq 4}^{0,4} \cong \ker d_3 \cong 0$, so $H^5(K(\mathbb{Z},3);\mathbb{Z}) \cong E^{5,0} \cong 0$.

(iv) If $H^6(K(\mathbb{Z},3);\mathbb{Z}) \cong 0$, then we would have $\ker(d_3 : E_3^{3,2} \to E_3^{6,0}) = E_3^{3,2}$. We have already seen that $\operatorname{im}(d_3 : E_3^{0,4} \to E_3^{3,2}) = 2x\iota$, so it would follow that $E_4^{3,2}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ (generated by $x\iota$). But now there are no other differentials hitting $E_r^{3,2}$, which would imply that this $\mathbb{Z}/2\mathbb{Z}$ survives to $H^5(*;\mathbb{Z})$.

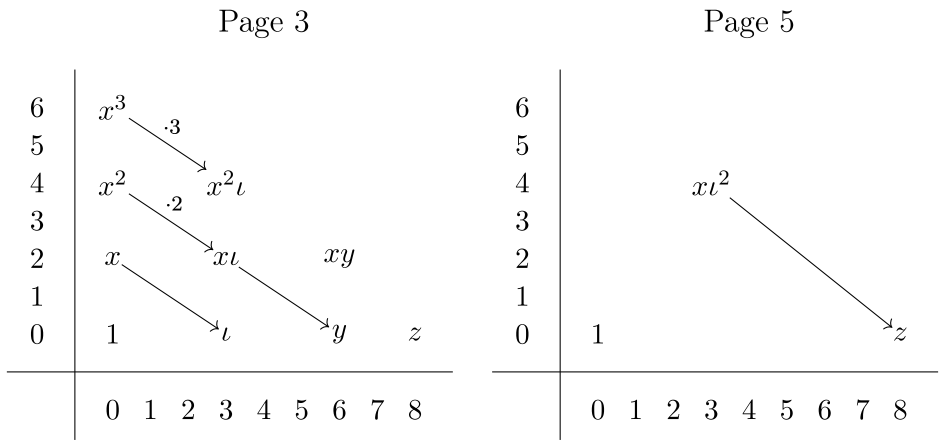


FIGURE 3. The $\mathbb{CP}^\infty \to * \to K(\mathbb{Z}, 3)$ spectral sequence

By contradiction, we deduce that $y := d_3(x\iota) \in H^6(K(\mathbb{Z}, 3); \mathbb{Z})$ is non-zero. But $2y = d_3(2x\iota) = 0$, as $2x\iota = d_3(x^2)$. Thus $E_3^{6,0}$ contains a $\mathbb{Z}/2\mathbb{Z}$ subgroup, and $E_4^{6,0}$ is the quotient by this $\mathbb{Z}/2\mathbb{Z}$. There are no other differentials with non-zero domain hitting $E^{6,0}$, so we conclude that $E_4^{6,0} \cong 0$ and thus $E_3^{6,0} \cong H^6(K(\mathbb{Z}, 3); \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.

(v) Degree 7 is analogous to degree 5. The only possible non-zero differential is $d_7 : E_7^{0,6} \to E_7^{7,0}$, but $E_3^{0,6}$ is generated by x^3. Since $d_3(x^3) = 3x^2\iota$ is non-zero and non-torsion, we find that $E_{\geq 4}^{0,6} \cong \ker d_3 \cong 0$ and hence $H^7(K(\mathbb{Z}, 3); \mathbb{Z}) \cong E^{7,0} \cong 0$.

(vi) Consider $x^2\iota \in E_3^{3,4}$. Using the product rule, we have

$$\begin{aligned} d_3(x^2\iota) &= x d_3(x\iota) + d_3(x)x\iota \\ &= xy + x\iota^2 \in E_3^{6,2}. \end{aligned}$$

Since $3x^2\iota = d_3(x^3)$, we see that $3d_3(x^2\iota) = 3xy + 3x\iota^2 = 0$. Our previous computations and the ring structure on $E^{p,q}$ imply that $E_3^{6,2}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ with generator xy, so $x\iota^2 = xy$ and $d_3(x^2\iota) = 0$. It follows that $E_4^{3,4} \cong \mathbb{Z}/3\mathbb{Z}$ with generator $x^2\iota$.

The only other possible non-zero differential out of $E_2^{3,4}$ is $d_5 : E_5^{3,4} \to E_5^{8,0}$. There are no differentials into $E_5^{3,4}$, so d_5 is injective. Moreover, there are no other differentials into $E_5^{8,0}$, so d_5 is an isomorphism. Thus $E_5^{8,0} \cong H^8(K(\mathbb{Z}, 3); \mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$ (whose generator is denoted by z in Figure 3). □

The third fact we need is that the second Pontryagin class of the canonical bundle $\gamma \to \mathrm{BO}\langle 8\rangle$ is $\pm 6 \in H^8(\mathrm{BO}\langle 8\rangle; \mathbb{Z})$.

Lemma 5.8. *Let γ be the canonical bundle on* $\mathrm{BO}\langle 8\rangle$*. Then there exists a generator $a \in H^8(\mathrm{BO}\langle 8\rangle; \mathbb{Z}) \cong \mathbb{Z}$ such that $p_2(\gamma) = 6a$, where p_2 is the second Pontryagin class.*

Proof. The space $\mathrm{BO}\langle 8\rangle$ can be obtained as the homotopy fiber of the map $\mathrm{BSpin} \to K(\mathbb{Z}, 4)$ inducing an isomorphism on π_4. Extending to the left, we get a fiber sequence of spaces

$$K(\mathbb{Z},3) \longrightarrow \mathrm{BO}\langle 8\rangle \longrightarrow \mathrm{BSpin}.$$

The Serre spectral sequence for this fibration has signature

$$E_2^{p,q} = H^p(\mathrm{BSpin}; H^q(K(\mathbb{Z},3);\mathbb{Z})) \Longrightarrow H^{p+q}(\mathrm{BO}\langle 8\rangle;\mathbb{Z})$$

(see Figure 4, in which $\bullet = \mathbb{Z}$, $\circ = \mathbb{Z}/2\mathbb{Z}$, and $\triangleleft = \mathbb{Z}/3\mathbb{Z}$). We computed $H^*(K(\mathbb{Z},3);\mathbb{Z})$ for $* \leq 8$ in Lemma 5.7, which gives us $E_2^{0,q}$ for $q \leq 8$. Let $\iota \in H^3(K(\mathbb{Z},3);\mathbb{Z})$ be a generator.

Next, we need to recall the integral cohomology BSpin in low degrees, which we can read out of [Dua19, Theorem 9.1]. These are given by

$$H^i(\mathrm{BSpin};\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i = 0,4 \\ 0 & i = 1,2,3,5,6, \\ \mathbb{Z}/2\mathbb{Z} & i = 7,9, \\ \mathbb{Z}^2 & i = 8. \end{cases} \tag{5.9}$$

In *loc. cit.*, it is also shown that there is a generator $\sigma_1 \in H^4(\mathrm{BSpin};\mathbb{Z})$ such that $2\sigma_1 = p_1(\beta)$ (the first Pontryagin class of the canonical bundle β on BSpin). Moreover, there is a class $\sigma_2 \in H^8(\mathrm{BSpin};\mathbb{Z})$ such that σ_1^2, σ_2 freely generate $H^8(\mathrm{BSpin};\mathbb{Z})$ and $\sigma_1^2 + 2\sigma_2 = p_2(\beta)$ (the second Pontryagin class).

Because $\mathrm{BO}\langle 8\rangle$ is 7-connected, its cohomology must vanish in degrees seven and below. The only way for $E_r^{0,3}$ and $E_r^{4,0}$ to die is if $d_4(\iota) = \pm\sigma_1$. Thus $d_4(\iota\sigma_1) = \pm\sigma_1^2$, so quotienting by this image leaves us with $E_5^{8,0} \cong \mathbb{Z}$, which is generated by σ_2. There are no other differentials into $E_r^{8,0}$, so we find that $H^8(\mathrm{BO}\langle 8\rangle;\mathbb{Z}) \cong \mathbb{Z}$ is generated by σ_2. Since $\sigma_1^2 = 0$ in this group, we have the relation $p_2(\gamma) = 2\sigma_2$.

By looking at the low degree cohomology groups of $K(\mathbb{Z},3)$ and BSpin, we see that the only other group of total degree 8 is $E_\infty^{0,8} \cong H^8(K(\mathbb{Z},3);\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$. The only possible non-zero differential out of $E^{0,8}$ is $d_9 : E_9^{0,8} \to E_9^{9,0} \cong H^9(\mathrm{BSpin};\mathbb{Z})$. However, this differential must be zero, because all the previous differentials hitting $E^{9,0}$ have trivial domain, and all torsion in $H^*(\mathrm{BSpin};\mathbb{Z})$ has order 2 [Sto68, p. 316]. By convergence of this spectral sequence, there is a subgroup $A \subseteq H^8(\mathrm{BO}\langle 8\rangle;\mathbb{Z}) \cong \mathbb{Z}$ such that $E_\infty^{8,0} \cong A$ and $E_\infty^{0,8} \cong H^8(\mathrm{BO}\langle 8\rangle;\mathbb{Z})/A$. Thus A can be identified with $3\mathbb{Z} \subset \mathbb{Z} \cong H^8(\mathrm{BO}\langle 8\rangle;\mathbb{Z})$. In particular, $\sigma_2 = 3a$ for some generator $a \in H^8(\mathrm{BSpin};\mathbb{Z})$, so $p_2(\gamma) = 2\sigma_2 = 6a$. $\square$

Finally, we need to know a little bit about the cohomology of the tensor product maps $\mathrm{BSO} \wedge \mathrm{BSpin} \to \mathrm{BO}\langle 8\rangle$.

Lemma 5.9. *Let γ be the classifying bundle on $\mathrm{BO}\langle 8\rangle$. Under the product map $\mathrm{BSO} \wedge \mathrm{BSpin} \to \mathrm{BO}\langle 8\rangle$, the image of a generator of a generator of $H^8(\mathrm{BO}\langle 8\rangle;\mathbb{Z}) \cong \mathbb{Z}$ under the induced map*

$$H^8(\mathrm{BO}\langle 8\rangle;\mathbb{Z}) \to H^8(\mathrm{BSO} \wedge \mathrm{BSpin};\mathbb{Z})$$

is of the form $2s+t$, where t is a torsion class and the class s has the property that $s+t'$ is not divisible by 2 for any torsion class t'.

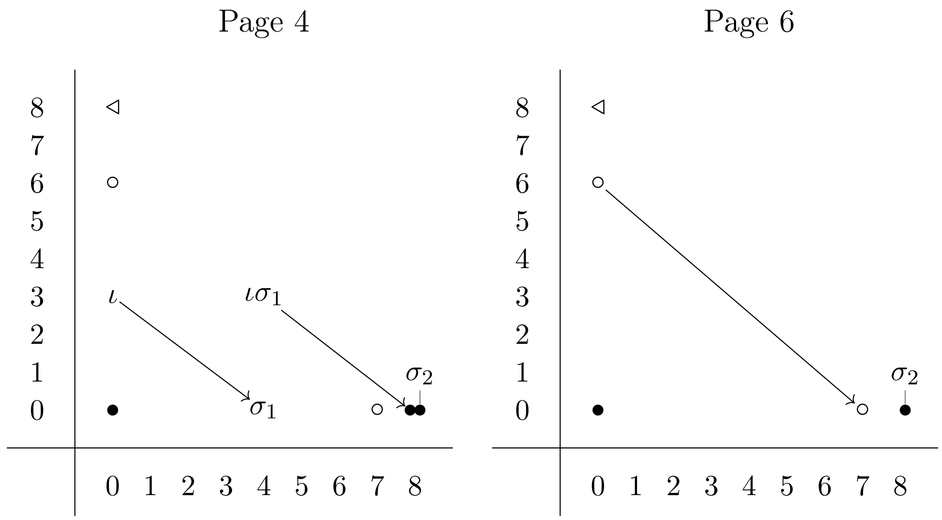


FIGURE 4. The $K(\mathbb{Z},3) \to \mathrm{BO}\langle 8\rangle \to \mathrm{BSpin}$ spectral sequence

Proof. Let α and β be the classifying bundles on BSO, and BSpin, respectively. Recall that $H^i(\mathrm{BSO};\mathbb{Z}) \cong 0$ for $1 \leq i \leq 3$ and $H^4(\mathrm{BSO};\mathbb{Z}) \cong \mathbb{Z}$, generated by the first Pontryagin class $p_1(\alpha)$ [BJ82]. Also $H^i(\mathrm{BSpin};\mathbb{Z}) \cong 0$ for $1 \leq i \leq 3$ and $H^4(\mathrm{BSpin};\mathbb{Z}) \cong \mathbb{Z}$ with generator σ_1 satisfying $2\sigma_1 = p_1(\beta)$ [Dua19, Theorem 9.1]. The product map $\mathrm{BSO} \wedge \mathrm{BSpin} \to \mathrm{BO}\langle 8\rangle$ has the class $c_4(\gamma_{\mathbb{C}})$ mapping to $c_4(\alpha_{\mathbb{C}} \otimes \beta_{\mathbb{C}})$ in cohomology because Chern classes are natural, and this is equal to $-6c_2(\alpha_{\mathbb{C}})c_2(\beta_{\mathbb{C}})$ by Lemma 5.6, as $H^2(\mathrm{BSO};\mathbb{Z}) \cong H^2(\mathrm{BSpin};\mathbb{Z}) \cong 0$ which forces the first Chern classes to vanish for $\alpha_{\mathbb{C}}$ and $\beta_{\mathbb{C}}$. Since $c_2(\alpha_{\mathbb{C}}) = -p_1(\alpha)$ and $c_2(\beta_{\mathbb{C}}) = -p_1(\beta) = -2\sigma_1$, we find that $c_4(\gamma_{\mathbb{C}})$ maps to $-12p_1(\alpha)\sigma_1$.

Let $a \in H^8(\mathrm{BO}\langle 8\rangle;\mathbb{Z})$ be the generator such that $c_4(\gamma_{\mathbb{C}}) = p_2(\gamma)$ is equal to $6a$, as given by Lemma 5.8. Then a maps to $-2p_1(\alpha)\sigma_1 + t$ under $H^*(\mathrm{BO}\langle 8\rangle;\mathbb{Z}) \to H^*(\mathrm{BSO} \wedge \mathrm{BSpin};\mathbb{Z})$, where $t \in H^8(\mathrm{BSO} \wedge \mathrm{BSpin};\mathbb{Z})$ is some element satisfying $6t = 0$.

It thus remains to show that $s := -p_1(\alpha)\sigma_1$ is such that $s + t'$ is not divisble by two for any torsion class t'. Using the Künneth formula for cohomology (see [Dol72, Proposition VI.12.16]), we get a split short exact sequence

$$0 \longrightarrow H^4(\mathrm{BSO};\mathbb{Z}) \otimes H^4(\mathrm{BSpin};\mathbb{Z}) \longrightarrow H^8(\mathrm{BSO} \wedge \mathrm{BSpin};\mathbb{Z}) \longrightarrow A \longrightarrow 0$$

where A is some Tor term. So we have a direct sum decomposition

$$H^8(\mathrm{BSO} \wedge \mathrm{BSpin};\mathbb{Z}) \cong (H^4(\mathrm{BSO};\mathbb{Z}) \otimes H^4(\mathrm{BSpin};\mathbb{Z})) \oplus A.$$

We know that s belongs to the first summand because it is a product of a class of BSO and a class of BSpin. Also t' must belong to A since it is a torsion class and $H^4(\mathrm{BSO};\mathbb{Z}) \otimes H^4(\mathrm{BSpin};\mathbb{Z}) \cong \mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}$ is torsion-free. Since s is a generator of $H^4(\mathrm{BSO};\mathbb{Z}) \otimes H^4(\mathrm{BSpin};\mathbb{Z})$ and t' lives in the other summand, $s + t'$ cannot be a multiple of 2. $\square$

5.4. **Module structure for suspended elephants.** We are now ready to return to our goal of lifting the multiplication $\mathrm{ko}\langle 8k+2\rangle \wedge \mathrm{ksp} \to \mathrm{ksp}\langle 8k+2\rangle$ to $\Sigma^{8k+4}F$. We will first show that such a lift exists, after which we will compute its effect on cohomology.

Proposition 5.10. *The multiplication map* $\mathrm{ko}\langle 8k+2\rangle \wedge \mathrm{ksp} \to \mathrm{ksp}\langle 8k+2\rangle$ *lifts to a multiplication map* $\mathrm{ko}\langle 8k+2\rangle \wedge \mathrm{ksp} \to \Sigma^{8k+4}F$.

Proof. Since $\pi_{8k+2}\mathrm{KSp} \cong \pi_{8k+3}\mathrm{KSp} \cong 0$, we have $\mathrm{ksp}\langle 8k+2\rangle \cong \mathrm{ksp}\langle 8k+4\rangle$. Thus the map $\mathrm{ko}\langle 8k+2\rangle \wedge \mathrm{ksp} \to \mathrm{ksp}\langle 8k+2\rangle$ induces a map $\mathrm{ko}\langle 8k+2\rangle \wedge \mathrm{ksp} \to \mathrm{ksp}\langle 8k+4\rangle$. On cohomology, this map is determined by the image of y_{8k+4}, which generates $H^*\mathrm{ksp}\langle 8k+4\rangle$ (see Corollary 4.8). The action of $\mathcal{A}_1$ on y_{8k+4} is trivial, since $\mathrm{Sq}^1 y_{8k+4} = \mathrm{Sq}^2 y_{8k+4} = 0$.

Proposition 4.7 and Corollary 4.8 imply that $H^{8k+4}(\mathrm{ko}\langle 8k+2\rangle \wedge \mathrm{ksp})$ is generated by $\mathrm{Sq}^2 x_{8k+2} \otimes y_0$ and $x_{8k+2} \otimes \mathrm{Sq}^2 y_0$, since the cohomology over a field of a smash product is the tensor product of the cohomology of its factors, and the Steenrod algebra acts by the Cartan formula. But the action of $\mathcal{A}_1$ on any non-zero combination of these generators is non-trivial, since

$$\begin{aligned}
\mathrm{Sq}^1(\mathrm{Sq}^2 x_{8k+2} \otimes y_0) &= 0,\\
\mathrm{Sq}^2(\mathrm{Sq}^2 x_{8k+2} \otimes y_0) &= \mathrm{Sq}^3\mathrm{Sq}^1 x_{8k+2} \otimes y_0 + \mathrm{Sq}^2 x_{8k+2} \otimes \mathrm{Sq}^2 y_0,\\
\mathrm{Sq}^1(x_{8k+2} \otimes \mathrm{Sq}^2 y_0) &= \mathrm{Sq}^1 x_{8k+2} \otimes \mathrm{Sq}^2 y_0 + x_{8k+2} \otimes \mathrm{Sq}^3 y_0,\\
\mathrm{Sq}^2(x_{8k+2} \otimes \mathrm{Sq}^2 y_0) &= \mathrm{Sq}^2 x_{8k+2} \otimes \mathrm{Sq}^2 y_0 + \mathrm{Sq}^1 x_{8k+2} \otimes \mathrm{Sq}^3 y_0.
\end{aligned}$$

The action of $\mathcal{A}_1$ on the image of y_{8k+4} must be trivial, so we deduce that $H^*\mathrm{ksp}\langle 8k+4\rangle \to H^*(\mathrm{ko}\langle 8k+2\rangle \wedge \mathrm{ksp})$ is given by $y_{8k+4} \mapsto 0$.

Due to the fiber sequence given in Equation 5.4, the map $\mathrm{ko}\langle 8k+2\rangle \wedge \mathrm{ksp} \to \mathrm{ksp}\langle 8k+4\rangle$ lifts to a map $\mathrm{ko}\langle 8k+2\rangle \wedge \mathrm{ksp} \to \Sigma^{8k+4}F$ if the composite

$$\mathrm{ko}\langle 8k+2\rangle \wedge \mathrm{ksp} \to \mathrm{ksp}\langle 8k+4\rangle \to \Sigma^{8k+4}H\mathbb{Z}/2\mathbb{Z}$$

is nullhomotopic. Since the map $\mathrm{ksp}\langle 8k+4\rangle \to \Sigma^{8k+4}H\mathbb{Z}/2\mathbb{Z}$ represents y_{8k+4} and y_{8k+4} maps to zero in the cohomology of $\mathrm{ko}\langle 8k+2\rangle \wedge \mathrm{ksp}$, it follows that this map is nullhomotopic and we get a lift. □

Our next objective is to understand what the map $\mathrm{ko}\langle 8k+2\rangle \wedge \mathrm{ksp} \to \Sigma^{8k+4}F$ does in cohomology. After introducing some notation, we will study this map for $k = 0$.

Notation 5.11. Recall from Lemma 5.4 that $H^*F \cong \mathcal{A} \otimes_{\mathcal{A}_1} E$ is generated by Sq^1 and Sq^2. We denote these by $e_0 := \mathrm{Sq}^1$ and $e_1 := \mathrm{Sq}^2$.

Lemma 5.12. *Given any lift* $\mathrm{ko}\langle 2\rangle \wedge \mathrm{ko}\langle 4\rangle \to \Sigma^8 F$ *of the multiplication map* $\mathrm{ko}\langle 2\rangle \wedge \mathrm{ko}\langle 4\rangle \to \mathrm{ko}\langle 8\rangle$, *the image of* $e_0 \in H^8\Sigma^8 F$ *under the induced map* $H^8\Sigma^8 F \to H^8(\mathrm{ko}\langle 2\rangle \wedge \mathrm{ko}\langle 4\rangle)$ *is non-zero.*

Proof. Recall that if $\Phi \to E \to B$ is a fibration of spaces, where Φ is m-connected and B is n-connected, then there is a Serre exact sequence

$$0 \to H^0(B;\mathbb{Z}) \to H^0(E;\mathbb{Z}) \to H^0(\Phi;\mathbb{Z}) \to \cdots \to H^{m+n+1}(B;\mathbb{Z}).$$

In particular, if B is n-connected, then ΩB is $(n-1)$-connected, so the path-loop fibration gives us maps $H^i(\Omega B;\mathbb{Z}) \to H^{i+1}(B;\mathbb{Z})$ that are isomorphisms for $i < 2n-1$ and an injection for $i = 2n-1$.

Now if X is a CW spectrum, we can write X as the union of the subspectra generated by each level X_n. By [Ada74, Part III, Proposition 8.1], we can thus compute $H^*(X;\mathbb{Z})$ via the Milnor exact sequence

$$0 \to \lim{}^1 H^*(X_n;\mathbb{Z}) \to H^*(X;\mathbb{Z}) \to \lim H^*(X_n;\mathbb{Z}) \to 0.$$

So if X is a connective Ω-spectrum with n-connected X_0, then $H^i(X_0;\mathbb{Z}) \to H^i(X;\mathbb{Z})$ is an isomorphism for $i \le 2n$ by the Serre exact sequence above, as the connectivity of each loop space will always be at least n, so the maps making up the limit are all isomorphisms in this range by the Serre exact sequence. The zeroth spaces of $\mathrm{ko}\langle 8\rangle$, $\mathrm{ko}\langle 4\rangle$, and $\mathrm{ko}\langle 2\rangle$ are $\mathrm{BO}\langle 8\rangle$, BSpin and BSO, respectively. Since $\mathrm{BO}\langle 8\rangle$ is 7-connected (by construction) and BSpin is 3-connected (as Spin is 2-connected), we have isomorphisms

$$\begin{aligned} H^i(\mathrm{ko}\langle 8\rangle;\mathbb{Z}) &\cong H^i(\mathrm{BO}\langle 8\rangle;\mathbb{Z}) && (\text{for } i \le 14),\\ H^j(\mathrm{ko}\langle 4\rangle;\mathbb{Z}) &\cong H^j(\mathrm{BSpin};\mathbb{Z}) && (\text{for } j \le 6).\end{aligned}$$

For BSO and $\mathrm{ko}\langle 2\rangle$ we need more care because BSO is only 1-connected. However, since the integral cohomology of BSO is trivial in degrees less than four, if Φ is a delooping of BSO, we have a fiber sequence $\Phi \to * \to \mathrm{BSO}$. The Serre spectral sequence of this fiber sequence implies that the integral cohomology of Φ is trivial for degrees less than five and the transgression $d_5 : E_5^{0,4} \to E_5^{5,0}$ is an isomorphism. Similarly, for any higher delooping of BSO, the transgression must be an isomorphism on these bottom cohomology groups for the same reason, and therefore

$$H^j(\mathrm{ko}\langle 2\rangle;\mathbb{Z}) \cong H^j(\mathrm{BSO};\mathbb{Z}) \qquad (\text{for } j \le 4).$$

It follows that the generators

$$\begin{aligned} a &\in H^8(\mathrm{BO}\langle 8\rangle;\mathbb{Z}),\\ \sigma_1 &\in H^4(\mathrm{BSpin};\mathbb{Z}),\\ p_1(\alpha) &\in H^4(\mathrm{BSO};\mathbb{Z})\end{aligned}$$

determine generators in $H^8(\mathrm{ko}\langle 8\rangle;\mathbb{Z})$, $H^4(\mathrm{ko}\langle 4\rangle;\mathbb{Z})$, and $H^4(\mathrm{ko}\langle 2\rangle;\mathbb{Z})$, respectively.

Using one of Adams's models of spectra and smash products [Ada74], the zeroth space of $\mathrm{ko}\langle 2\rangle \wedge \mathrm{ko}\langle 4\rangle$ is $\mathrm{BSO} \wedge \mathrm{BSpin}$, which is at least $(1+3+1)$-connected. Thus $H^i(\mathrm{ko}\langle 2\rangle \wedge \mathrm{ko}\langle 4\rangle;\mathbb{Z}) \cong H^i(\mathrm{BSO} \wedge \mathrm{BSpin};\mathbb{Z})$ for $i \le 10$. It now follows from Lemma 5.9 that any generator of $H^*(\mathrm{ko}\langle 8\rangle;\mathbb{Z})$ is sent to an element of the form $2s + t \in H^8(\mathrm{ko}\langle 2\rangle \wedge \mathrm{ko}\langle 4\rangle;\mathbb{Z})$, where t is a torsion class and s has the property that $s + t'$ is not a multiple of two for any torsion class t'.

Finally, either generator of $\pi_8\Sigma^8 F \cong \mathbb{Z}$ is sent to twice a generator of $\pi_8\mathrm{ko}\langle 8\rangle \cong \mathbb{Z}$ under the map induced by $\Sigma^8 F \to \mathrm{ko}\langle 8\rangle$ since we defined $F \to \mathrm{ko}$ to be the inclusion $2\mathbb{Z} \to \mathbb{Z}$. So the Hurewicz theorem implies that in homology, either generator of $H_8(\Sigma^8 F;\mathbb{Z}) \cong \mathbb{Z}$ maps to twice a generator of $H_8(\mathrm{ko}\langle 8\rangle;\mathbb{Z}) \cong \mathbb{Z}$. By the universal coefficient theorem, either generator of $H^8(\mathrm{ko}\langle 8\rangle;\mathbb{Z}) \cong \mathbb{Z}$ goes to twice a generator of $H^8(\Sigma^8 F;\mathbb{Z})$. By assumption, the map $\mathrm{ko}\langle 2\rangle \wedge \mathrm{ko}\langle 4\rangle \to \mathrm{ko}\langle 8\rangle$ factors as

$$\begin{tikzcd} & \Sigma^8 F \arrow[d] \\ \mathrm{ko}\langle 2\rangle \wedge \mathrm{ko}\langle 4\rangle \arrow[ur] \arrow[r] & \mathrm{ko}\langle 8\rangle, \end{tikzcd}$$

so the map $H^8(\mathrm{ko}\langle 8\rangle;\mathbb{Z}) \to H^8(\mathrm{ko}\langle 2\rangle \wedge \mathrm{ko}\langle 4\rangle;\mathbb{Z})$ must factor through $H^8(\Sigma^8 F;\mathbb{Z}) \to H^8(\mathrm{ko}\langle 2\rangle \wedge \mathrm{ko}\langle 4\rangle;\mathbb{Z})$. So if $a \in H^8(\mathrm{ko}\langle 8\rangle;\mathbb{Z})$ is a generator, then we have

$$\begin{array}{ccccc} H^8(\mathrm{ko}\langle 8\rangle;\mathbb{Z}) & \longrightarrow & H^8(\Sigma^8 F;\mathbb{Z}) & \longrightarrow & H^8(\mathrm{ko}\langle 2\rangle \wedge \mathrm{ko}\langle 4\rangle;\mathbb{Z}) \\ a & \longmapsto & 2b & \longmapsto & 2s+t, \end{array}$$

where $b \in H^8(\Sigma^8 F;\mathbb{Z})$ is a generator. Thus $b \mapsto s + t'$, where s is not divisible by 2 and t' is a torsion class. The mod 2 reduction of b is $e_0 \in H^8\Sigma^8 F$, and the mod 2 reduction of $s + t'$ is a non-zero element of $H^8(\mathrm{ko}\langle 2\rangle \wedge \mathrm{ko}\langle 4\rangle)$ since it is not a multiple of two. $\square$

Now that we know that the image of e_0 is non-zero, we can explicitly determine what value this image takes.

Lemma 5.13. *Any lift $\mathrm{ko}\langle 2\rangle \wedge \mathrm{ko}\langle 4\rangle \to \Sigma^8 F$ induces $e_0 \mapsto \mathrm{Sq}^2 x_2 \otimes x_4$ in cohomology.*

Proof. Since $H^*\mathrm{ko}\langle 2\rangle \cong \mathcal{A}/\mathcal{A}\mathrm{Sq}^3$ and $H^*\mathrm{ko}\langle 4\rangle \cong \mathcal{A}/(\mathcal{A}\mathrm{Sq}^1 + \mathcal{A}\mathrm{Sq}^5)$, we can read off the possible images that e_0 might have. The $\mathcal{A}$-module $H^*\mathrm{ko}\langle 2\rangle$ is generated by x_2 in degree two, $\mathrm{Sq}^1 x_2$ in degree three, and $\mathrm{Sq}^2 x_2$ in degree four, and $H^*\mathrm{ko}\langle 4\rangle$ has x_4 in degree four, nothing in degree five, and $\mathrm{Sq}^2 x_4$ in degree six.

So we know that $e_0 \mapsto A\mathrm{Sq}^2 x_2 \otimes x_4 + Bx_2 \otimes \mathrm{Sq}^2 x_4$ for some $A, B \in \mathbb{Z}/2\mathbb{Z}$ with $(A, B) \neq (0, 0)$. However, note that $\mathrm{Sq}^1 e_0 = 0$ and

$$\begin{aligned} &\mathrm{Sq}^1(A\mathrm{Sq}^2 x_2 \otimes x_4 + Bx_2 \otimes \mathrm{Sq}^2 x_4) \\ = {}& A\mathrm{Sq}^3 x_2 \otimes x_4 + A\mathrm{Sq}^2 x_2 \otimes \mathrm{Sq}^1 x_4 + B\mathrm{Sq}^1 x_2 \otimes \mathrm{Sq}^2 x_4 + Bx_2 \otimes \mathrm{Sq}^3 x_4 \\ = {}& B\mathrm{Sq}^1 x_2 \otimes \mathrm{Sq}^2 x_4 + Bx_2 \otimes \mathrm{Sq}^3 x_4, \end{aligned}$$

This is only zero if B is zero, so $H^*\Sigma^8 F \to H^*(\mathrm{ko}\langle 2\rangle \wedge \mathrm{ko}\langle 4\rangle)$ is a homomorphism of $\mathcal{A}$-modules if and only if $A = 1$ and $B = 0$. Thus $e_0 \mapsto \mathrm{Sq}^2 x_2 \otimes x_4$. $\square$

So far, we know that $\mathrm{ko}\langle 2\rangle \wedge \mathrm{ko}\langle 4\rangle \to \mathrm{ko}\langle 8\rangle$ lifts to $\mathrm{ko}\langle 2\rangle \wedge \mathrm{ko}\langle 4\rangle \to \Sigma^8 F$, and we know the image of $e_0 \in H^8\Sigma^8 F$ under any such lift. Next, we show that there is a lift such that $e_1 \mapsto x_2 \otimes \mathrm{Sq}^3 x_4$. This is a key property that the maps in the splitting must have in order to get an isomorphism in cohomology with $\mathbb{Z}/2\mathbb{Z}$ coefficients later.

Lemma 5.14. *There exists a lift* $\mathrm{ko}\langle 2\rangle \wedge \mathrm{ko}\langle 4\rangle \to \Sigma^8 F$ *such that* $e_1 \mapsto x_2 \otimes \mathrm{Sq}^3 x_4$.

Proof. We use obstruction theory to obtain a lift with the desired properties. Since $H^*\mathrm{ko}\langle 2\rangle$ has $\mathrm{Sq}^2\mathrm{Sq}^1 x_2$ in degree five and $H^*\mathrm{ko}\langle 4\rangle$ has $\mathrm{Sq}^3 x_4$ in degree seven, we have

$$e_1 \mapsto C\mathrm{Sq}^2\mathrm{Sq}^1 x_2 \otimes x_4 + D\mathrm{Sq}^1 x_2 \otimes \mathrm{Sq}^2 x_4 + E x_2 \otimes \mathrm{Sq}^3 x_4$$

for $C, D, E \in \mathbb{Z}/2\mathbb{Z}$. We can eliminate some possibilities using the relation $\mathrm{Sq}^2 e_1 = \mathrm{Sq}^3 e_0$. On the right hand side, Lemma 5.13 implies that

$$\begin{aligned}
&\mathrm{Sq}^3(\mathrm{Sq}^2 x_2 \otimes x_4) \\
&= \mathrm{Sq}^3\mathrm{Sq}^2 x_2 \otimes x_4 + \mathrm{Sq}^2\mathrm{Sq}^2 x_2 \otimes \mathrm{Sq}^1 x_4 + \mathrm{Sq}^1\mathrm{Sq}^2 x_2 \otimes \mathrm{Sq}^2 x_4 + \mathrm{Sq}^2 x_2 \otimes \mathrm{Sq}^3 x_4 \\
&= \mathrm{Sq}^3 x_2 \otimes \mathrm{Sq}^2 x_4 + \mathrm{Sq}^2 x_2 \otimes \mathrm{Sq}^3 x_4 \\
&= \mathrm{Sq}^2 x_2 \otimes \mathrm{Sq}^3 x_4.
\end{aligned}$$

For the left hand side, we compute

$$\begin{aligned}
&\mathrm{Sq}^2(C\mathrm{Sq}^2\mathrm{Sq}^1 x_2 \otimes x_4 + D\mathrm{Sq}^1 x_2 \otimes \mathrm{Sq}^2 x_4 + E x_2 \otimes \mathrm{Sq}^3 x_4) \\
&= C\mathrm{Sq}^2\mathrm{Sq}^2\mathrm{Sq}^1 x_2 \otimes x_4 + C\mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^1 x_2 \otimes \mathrm{Sq}^1 x_4 + C\mathrm{Sq}^2\mathrm{Sq}^1 x_2 \otimes \mathrm{Sq}^2 x_4 \\
&\quad + D\mathrm{Sq}^2\mathrm{Sq}^1 x_2 \otimes \mathrm{Sq}^2 x_4 + D\mathrm{Sq}^1\mathrm{Sq}^1 x_2 \otimes \mathrm{Sq}^1\mathrm{Sq}^2 x_4 + D\mathrm{Sq}^1 x_2 \otimes \mathrm{Sq}^2\mathrm{Sq}^2 x_4 \\
&\quad + E\mathrm{Sq}^2 x_2 \otimes \mathrm{Sq}^3 x_4 + E\mathrm{Sq}^1 x_2 \otimes \mathrm{Sq}^1\mathrm{Sq}^3 x_4 + E x_2 \otimes \mathrm{Sq}^2\mathrm{Sq}^3 x_4 \\
&= C\mathrm{Sq}^2\mathrm{Sq}^1 x_2 \otimes \mathrm{Sq}^2 x_4 + D\mathrm{Sq}^2\mathrm{Sq}^1 x_2 \otimes \mathrm{Sq}^2 x_4 + E\mathrm{Sq}^2 x_2 \otimes \mathrm{Sq}^3 x_4.
\end{aligned}$$

In order for this to equal $\mathrm{Sq}^2 x_2 \otimes \mathrm{Sq}^3 x_4$, we must have $E = 1$ and either $C = D = 0$ or $C = D = 1$. We are done if $C = D = 0$, so we may assume $C = D = 1$. Let $f : \mathrm{ko}\langle 2\rangle \wedge \mathrm{ko}\langle 4\rangle \to \Sigma^8 F$ be the lift inducing

$$e_1 \mapsto \mathrm{Sq}^2\mathrm{Sq}^1 x_2 \otimes x_4 + \mathrm{Sq}^1 x_2 \otimes \mathrm{Sq}^2 x_4 + x_2 \otimes \mathrm{Sq}^3 x_4.$$

Rotating the fiber sequence given in Equation 5.4, we see that there is a fiber sequence

$$\Sigma^7 H\mathbb{Z}/2\mathbb{Z} \longrightarrow \Sigma^8 F \longrightarrow \mathrm{ko}\langle 8\rangle. \tag{5.10}$$

The image of $H^*\mathrm{ko}\langle 8\rangle \to H^*\Sigma^8 F$ is zero by Proposition 4.7 and Lemma 5.4, so the long exact sequence associated to Equation 5.10 implies that the map $H^*\Sigma^8 F \to H^*\Sigma^7 H\mathbb{Z}/2\mathbb{Z}$ is injective. This forces the generators e_0 and e_1 to map to Sq^1 and Sq^2, respectively, and hence completely determines the map $H^*\Sigma^8 F \to H^*\Sigma^7 H\mathbb{Z}/2\mathbb{Z}$.

Now consider the map $\mathrm{ko}\langle 2\rangle \wedge \mathrm{ko}\langle 4\rangle \to \Sigma^7 H\mathbb{Z}/2\mathbb{Z}$ classifying $\mathrm{Sq}^1 x_2 \otimes x_4 \in H^7(\mathrm{ko}\langle 2\rangle \wedge \mathrm{ko}\langle 4\rangle)$. Composing with the map $\Sigma^7 H\mathbb{Z}/2\mathbb{Z} \to \Sigma^8 F$ from Equation 5.10, we get a map

$g : \mathrm{ko}\langle 2\rangle \wedge \mathrm{ko}\langle 4\rangle \to \Sigma^8 F$ such that

$$\begin{aligned}
g^* e_0 &= \mathrm{Sq}^1(\mathrm{Sq}^1 x_2 \otimes x_4) \\
&= \mathrm{Sq}^1\mathrm{Sq}^1 x_2 \otimes x_4 + \mathrm{Sq}^1 x_2 \otimes \mathrm{Sq}^1 x_4 \\
&= 0, \\
g^* e_1 &= \mathrm{Sq}^2(\mathrm{Sq}^1 x_2 \otimes x_4) \\
&= \mathrm{Sq}^2\mathrm{Sq}^1 x_2 \otimes x_4 + \mathrm{Sq}^1\mathrm{Sq}^1 x_2 \otimes \mathrm{Sq}^1 x_4 + \mathrm{Sq}^1 x_2 \otimes \mathrm{Sq}^2 x_4 \\
&= \mathrm{Sq}^2\mathrm{Sq}^1 x_2 \otimes x_4 + \mathrm{Sq}^1 x_2 \otimes \mathrm{Sq}^2 x_4.
\end{aligned}$$

Since the composite

$$\mathrm{ko}\langle 2\rangle \wedge \mathrm{ko}\langle 4\rangle \xrightarrow{g} \Sigma^8 F \to \mathrm{ko}\langle 8\rangle \tag{5.11}$$

factors through the fiber sequence given in Equation 5.10 (by the definition of g), a nullhomotopy of the fiber sequence yields a nullhomotopy of Equation 5.11. Since $f : \mathrm{ko}\langle 2\rangle \wedge \mathrm{ko}\langle 4\rangle \to \Sigma^8 F$ is a lift of the product map $\mathrm{ko}\langle 2\rangle \wedge \mathrm{ko}\langle 4\rangle \to \mathrm{ko}\langle 8\rangle$, so is the sum $f + g : \mathrm{ko}\langle 2\rangle \wedge \mathrm{ko}\langle 4\rangle \to \Sigma^8 F$. In cohomology, we compute

$$\begin{aligned}
(f+g)^* e_0 &= f^* e_0 + g^* e_0 \\
&= \mathrm{Sq}^2 x_2 \otimes x_4 + 0 \\
&= \mathrm{Sq}^2 x_2 \otimes x_4, \\
(f+g)^* e_1 &= f^* e_1 + g^* e_1 \\
&= (\mathrm{Sq}^2\mathrm{Sq}^1 x_2 \otimes x_4 + \mathrm{Sq}^1 x_2 \otimes \mathrm{Sq}^2 x_4 + x_2 \otimes \mathrm{Sq}^3 x_4) \\
&\quad + (\mathrm{Sq}^2\mathrm{Sq}^1 x_2 \otimes x_4 + \mathrm{Sq}^1 x_2 \otimes \mathrm{Sq}^2 x_4) \\
&= x_2 \otimes \mathrm{Sq}^3 x_4.
\end{aligned}$$

Thus $f + g$ is the desired lift of $\mathrm{ko}\langle 2\rangle \wedge \mathrm{ko}\langle 4\rangle \to \mathrm{ko}\langle 8\rangle$. □

We now suspend this lift at $k = 0$ to obtain the desired lift for all k.

Lemma 5.15. *There is a lift* $\mathrm{ko}\langle 8k+2\rangle \wedge \mathrm{ksp} \to \Sigma^{8k+4} F$ *of the multiplication map* $\mathrm{ko}\langle 8k+2\rangle \wedge \mathrm{ksp} \to \mathrm{ksp}\langle 8k+2\rangle$ *such that* $e_0 \mapsto \mathrm{Sq}^2 x_{8k+2} \otimes y_0$ *and* $e_1 \mapsto x_{8k+2} \otimes \mathrm{Sq}^3 y_0$ *in cohomology.*

Proof. Consider the product map $\mathrm{ko}\langle 2\rangle \wedge \mathrm{ksp} \to \mathrm{ksp}\langle 4\rangle$, which is the unique top arrow making the square

$$\begin{array}{ccc}
\mathrm{ko}\langle 2\rangle \wedge \mathrm{ksp} & \longrightarrow & \mathrm{ksp}\langle 4\rangle \\
\big\downarrow & & \big\downarrow \\
\mathrm{KO} \wedge \mathrm{KSp} & \longrightarrow & \mathrm{KSp}
\end{array}$$

commute (using Diagram 5.1 and $\mathrm{ksp}\langle 2\rangle \cong \mathrm{ksp}\langle 4\rangle$). Suspending four times, we get the commutative diagram

$$\begin{tikzcd} \mathrm{ko}\langle 2\rangle \wedge \mathrm{ko}\langle 4\rangle \arrow[r] \arrow[d] & \mathrm{ko}\langle 8\rangle \arrow[d] \\ \mathrm{KO} \wedge \mathrm{KO} \arrow[r] & \mathrm{KO}. \end{tikzcd}$$

Using the isomorphism $\mathrm{KO} \cong \Sigma^4\mathrm{KSp}$ as KO-module spectra, the bottom arrow is the KO multiplication map. Thus the top arrow is the product map $\mathrm{ko}\langle 2\rangle \wedge \mathrm{ko}\langle 4\rangle \to \mathrm{ko}\langle 8\rangle$ appearing in Lemma 5.14. Now let

$$\begin{tikzcd} & \Sigma^8 F \arrow[d] \\ \mathrm{ko}\langle 2\rangle \wedge \mathrm{ko}\langle 4\rangle \arrow[r] \arrow[ur] & \mathrm{ko}\langle 8\rangle \end{tikzcd} \tag{5.12}$$

be a lift such that $e_0 \mapsto \mathrm{Sq}^2 x_2 \otimes x_4$ and $e_1 \mapsto x_2 \otimes \mathrm{Sq}^3 x_4$ in cohomology. Then the fourfold desuspension

$$\begin{tikzcd} & \Sigma^4 F \arrow[d] \\ \mathrm{ko}\langle 2\rangle \wedge \mathrm{ksp} \arrow[r] \arrow[ur] & \mathrm{ksp}\langle 4\rangle \end{tikzcd} \tag{5.13}$$

of Diagram 5.12 satisfies $e_0 \mapsto \mathrm{Sq}^2 x_2 \otimes y_0$ and $e_1 \mapsto x_2 \otimes \mathrm{Sq}^3 y_0$ in cohomology, since y_0 is the fourfold desuspension of x_4 (see Proposition 4.7 and Corollary 4.8). Now we suspend Diagram 5.13 another $8k$ times to get the diagram

$$\begin{tikzcd} & \Sigma^{8k+4} F \arrow[d] \\ \mathrm{ko}\langle 8k+2\rangle \wedge \mathrm{ksp} \arrow[r] \arrow[ur] \arrow[d] & \mathrm{ksp}\langle 8k+4\rangle \arrow[d] \\ \Sigma^{8k}\mathrm{KO} \wedge \mathrm{KSp} \arrow[r] & \Sigma^{8k}\mathrm{KSp}. \end{tikzcd}$$

Indeed, the bottom arrow is still the product map because $\Sigma^8\mathrm{KSp} \cong \mathrm{KSp}$ as KO-modules. So the top horizontal arrow is still the product. The map in cohomology induced by $\mathrm{ko}\langle 8k+2\rangle \wedge \mathrm{ksp} \to \Sigma^{8k+4}F$ is given by $e_0 \mapsto \mathrm{Sq}^2 x_{8k+2} \otimes y_0$ and $e_1 \mapsto x_{8k+2} \otimes \mathrm{Sq}^3 y_0$. $\square$

5.5. **Defining the** KSp**-Pontryagin and elephant classes.** Our next goal is to give maps $\mathrm{MSpin}^h \to \mathrm{ksp}\langle 8k\rangle$ and $\mathrm{MSpin}^h \to \Sigma^{8k+4}F$ that will constitute some of the summands in the 2-local splitting of MSpin^h. These maps arise from characteristic classes associated to $\mathrm{ksp}\langle 8k\rangle$ and $\Sigma^{8k+4}F$.

Setup 5.16. For each partition $I = (i_1, \dots, i_r)$, there is a KO-Pontryagin class $\pi_h^I \in \mathrm{KO}^0(\mathrm{BSpin}^h)$, obtained by pulling back the KO-Pontryagin class $\pi^I \in \mathrm{KO}^0(\mathrm{BSO})$ under

$\mathrm{BSpin}^h \to \mathrm{BSO}$. The associated map $\mathrm{BSpin}^h \to \mathrm{KO}$ admits a lift to $\mathrm{ko}\langle 4|I|\rangle$ if $I \in \mathcal{P}_{\text{even}}$ or to $\mathrm{ko}\langle 4|I|-2\rangle$ if $I \in \mathcal{P}_{\text{odd}}$ by Proposition 4.5.

Smashing with $\mathrm{MSpin}^h \xrightarrow{\varphi^h} \mathrm{KSp} \to \mathrm{ksp}$ (where φ^h is the Atiyah–Bott–Shapiro map and $\mathrm{KSp} \to \mathrm{ksp}$ is the canonical map to the connective cover), we get a map of the form

$$\mathrm{BSpin}^h \wedge \mathrm{MSpin}^h \to \mathrm{ko}\langle n\rangle \wedge \mathrm{ksp}, \tag{5.14}$$

where $n = 4|I|$ or $4|I|-2$ (depending on whether $|I|$ is even or odd). We now precompose Equation 5.14 with the Thom diagonal $\mathrm{MSpin}^h \to \mathrm{BSpin}^h \wedge \mathrm{MSpin}^h$ and postcompose with the multiplication $\mathrm{ko}\langle 4|I|\rangle \wedge \mathrm{ksp} \to \mathrm{ksp}\langle 4|I|\rangle$ (given in Lemma 5.1) or with the lift $\mathrm{ko}\langle 4|I|-2\rangle \wedge \mathrm{ksp} \to \Sigma^{4|I|}F$ (given in Lemma 5.15) of the multiplication $\mathrm{ko}\langle 4|I|-2\rangle \wedge \mathrm{ksp} \to \mathrm{ko}\langle 4|I|-2\rangle$.

When $I \in \mathcal{P}_{\text{even}}$, the composite takes the form

$$\mathrm{MSpin}^h \to \mathrm{BSpin}^h \wedge \mathrm{MSpin}^h \to \mathrm{ko}\langle 4|I|\rangle \wedge \mathrm{ksp} \to \mathrm{ksp}\langle 4|I|\rangle. \tag{5.15}$$

When $I \in \mathcal{P}_{\text{odd}}$, the composite takes the form

$$\mathrm{MSpin}^h \to \mathrm{BSpin}^h \wedge \mathrm{MSpin}^h \to \mathrm{ko}\langle 4|I|-2\rangle \wedge \mathrm{ksp} \to \Sigma^{4|I|}F. \tag{5.16}$$

Definition 5.17. Given an even partition I, the I^{th} KSp-*Pontryagin class* is the class $\kappa^I \in \mathrm{ksp}\langle 4|I|\rangle^0(\mathrm{MSpin}^h)$ determined by Equation 5.15. Given an odd partition I, the I^{th} *elephant class* is the class $\varepsilon^I \in \Sigma^{4|I|}F^0(\mathrm{MSpin}^h)$ determined by Equation 5.16. We refer to κ^I and ε^I collectively as KSp-*characteristic classes*.

Remark 5.18. When I is an odd partition, we still have a map

$$\mathrm{BSpin}^h \wedge \mathrm{MSpin}^h \to \mathrm{ko}\langle 4|I|\rangle \wedge \mathrm{ksp} \to \mathrm{ksp}\langle 4|I|\rangle$$

coming from Diagram 5.2. In particular, we have I^{th} KSp-Pontryagin classes κ^I for odd partitions as well, although we have not computed their effect on cohomology. These classes will not be needed for Theorem 1.1, but they will become relevant in Section 9.

We wish to compute the maps on cohomology induced by κ^I and ε^I. To this end, we need to compute $\mathrm{MSpin}^h \to \mathrm{ksp}$ in cohomology.

Lemma 5.19. *Let $\varphi^h : \mathrm{MSpin}^h \to \mathrm{KSp}$ be the Atiyah–Bott–Shapiro map. Then the composite $\mathrm{MSpin}^h \xrightarrow{\varphi^h} \mathrm{KSp} \to \mathrm{ksp}$ induces the map*

$$\begin{aligned} H^*\mathrm{ksp} &\to H^*\mathrm{MSpin}^h \\ y_0 &\mapsto U_h, \end{aligned}$$

where $U_h \in H^*\mathrm{MSpin}$ denotes the Thom class.

Proof. By [Hu22, Theorem 3.23], the induced map $\pi_0\mathrm{MSpin}^h \to \pi_0\mathrm{KSp}$ is surjective. But $\pi_0\mathrm{MSpin}^h \cong \mathbb{Z}$ (see [FH21, Theorem 9.97]) and $\pi_0\mathrm{KSp} \cong \mathbb{Z}$, so this must be an isomorphism. By the Hurewicz theorem, this means that $\mathrm{MSpin}^h \to \mathrm{ksp}$ must also give an isomorphism in degree zero integral homology, and then in degree zero mod

2 homology by reduction. Dualizing, we see that the map $\mathrm{MSpin}^h \to \mathrm{ksp}$ induces an isomorphism in mod 2 cohomology in degree zero, and therefore $y_0 \mapsto U_h$. ☐

Remark 5.20. The real and complex analogs of Lemma 5.19 can be proved by utilizing the fact that the real and complex Atiyah–Bott–Shapiro maps φ^r and φ^c are orientations and therefore preserve units. We have no such guarantee quaternionic Atiyah–Bott–Shapiro map φ^h, but it is plausible that the φ^r-module structure of φ^h enables a more conceptually parsimonious proof than the one we found.

Now we can compute the maps on cohomology induced by κ^I and ε^I.

Proposition 5.21. *Given a partition I, let $p_I \in H^*\mathrm{BSpin}^h$ denote the corresponding Pontryagin class. Let $U_h \in H^*\mathrm{MSpin}^h$ denote the Thom class. If $I \in \mathcal{P}_{\mathrm{even}}$, then the map $H^*\mathrm{ksp}\langle 4|I|\rangle \to H^*\mathrm{MSpin}^h$ induced by κ^I is given by*

$$y_{8k} \mapsto (p_I + \mathrm{Sq}^3\mathrm{Sq}^1\alpha_I)U_h,$$

where $\alpha_I \in H^*\mathrm{BSpin}^h$ is some class.

If $I \in \mathcal{P}_{\mathrm{odd}}$, then the map $H^*\Sigma^{4|I|}F \to H^*\mathrm{MSpin}^h$ induced by ε^I is given by

$$\begin{aligned} e_0 &\mapsto p_I U_h, \\ e_1 &\mapsto \beta_I w_3 U_h, \end{aligned}$$

where $\beta_I \in H^*\mathrm{BSpin}^h$ is some class satisfying $\mathrm{Sq}^2\beta_I = p_I$.

Proof. Recall that the Thom diagonal $\mathrm{MSpin}^h \to \mathrm{BSpin}^h \wedge \mathrm{MSpin}^h$ induces $a \otimes U_h \mapsto aU_h$ in cohomology. Since we have characterized the image of y_{8k} under $H^*\mathrm{ksp}\langle 4|I|\rangle \to H^*(\mathrm{ko}\langle 4|I|\rangle \wedge \mathrm{ksp})$ (Lemma 5.1) and the images of e_0 and e_1 under $H^*\Sigma^{4|I|}F \to H^*(\mathrm{ko}\langle 4|I| - 2\rangle \wedge \mathrm{ksp})$ (Lemma 5.15), it suffices to show that

$$\begin{aligned} H^*(\mathrm{ko}\langle 4|I|\rangle \wedge \mathrm{ksp}) &\to H^*(\mathrm{BSpin}^h \wedge \mathrm{MSpin}^h) \\ x_{4|I|} \otimes y_0 &\mapsto (p_I + \mathrm{Sq}^3\mathrm{Sq}^1\alpha_I) \otimes U_h \end{aligned}$$

for $I \in \mathcal{P}_{\mathrm{even}}$ and

$$\begin{aligned} H^*(\mathrm{ko}\langle 4|I| - 2\rangle \wedge \mathrm{ksp}) &\to H^*(\mathrm{BSpin}^h \wedge \mathrm{MSpin}^h) \\ \mathrm{Sq}^2 x_{4|I|-2} \otimes y_0 &\mapsto p_I \otimes U_h, \\ x_{4|I|-2} \otimes \mathrm{Sq}^3 y_0 &\mapsto \beta_I w_3 \otimes U_h \end{aligned}$$

for $I \in \mathcal{P}_{\mathrm{odd}}$. These are the maps induced by lifting KO-Pontryagin classes and smashing with the Atiyah–Bott–Shapiro map (see Setup 5.16). By [Sto68, p. 304], these lifts of KO-Pontryagin classes induce $x_{4|I|} \mapsto p_I + \delta\mathrm{Sq}^2\mathrm{Sq}^1\alpha_I$ and $\mathrm{Sq}^2 x_{4|I|-2} \mapsto p_I$ in integral cohomology,[4] where δ is the Bockstein. Let $\rho_2 : H^*(-;\mathbb{Z}) \to H^*(-;\mathbb{Z}/2\mathbb{Z})$ denote mod 2 reduction. Since $\rho_2 \circ \delta = \mathrm{Sq}^1$, we find that $\rho_2(\delta\mathrm{Sq}^2\mathrm{Sq}^1\alpha_I) = \mathrm{Sq}^3\mathrm{Sq}^1\alpha_I$. Thus $x_{4|I|} \mapsto p_I + \mathrm{Sq}^3\mathrm{Sq}^1\alpha_I$ for $I \in \mathcal{P}_{\mathrm{even}}$.

[4]We abuse notation by denoting integral classes and their mod 2 reductions by the same symbols.

For $I \in \mathcal{P}_{\text{odd}}$, we have $\mathrm{Sq}^2 x_{4|I|-2} \mapsto p_I$ in integral cohomology and hence $x_{4|I|-2} \mapsto \beta_I \in H^*\mathrm{BSpin}^h$. Lemma 5.19 states that $\mathrm{MSpin}^h \to \mathrm{ksp}$ induces $y_0 \mapsto U_h$ in cohomology, so we have $\mathrm{Sq}^3 y_0 \mapsto \mathrm{Sq}^3 U_h = w_3 U_h$ (by definition of the Stiefel–Whitney classes). Thus

$$\begin{aligned} x_{4|I|} \otimes y_0 &\mapsto (p_I + \mathrm{Sq}^3\mathrm{Sq}^1\alpha_I) \otimes U_h, \\ \mathrm{Sq}^2 x_{4|I|-2} \otimes y_0 &\mapsto p_I \otimes U_h, \\ x_{4|I|-2} \otimes \mathrm{Sq}^3 y_0 &\mapsto \beta_I \otimes \mathrm{Sq}^3 U_h = \beta_I w_3 \otimes U_h, \end{aligned}$$

as desired. □

Adding together the various KSp-Pontryagin classes and elephant classes gives us the first part of our eventual 2-local splitting of MSpin^h.

Proposition 5.22. *There exists a map*

$$\mathrm{MSpin}^h \to \bigvee_{I \in \mathcal{P}_{\text{even}}} \mathrm{ksp}\langle 4|I|\rangle \vee \bigvee_{I \in \mathcal{P}_{\text{odd}}} \Sigma^{4|I|} F$$

such that $y_{8k} \mapsto (p_I + \mathrm{Sq}^3\mathrm{Sq}^1\alpha_I)U_h$ *for some* $\alpha_I \in H^*\mathrm{BSpin}^h$ *(when* $I \in \mathcal{P}_{\text{even}}$*) and* $e_0 \mapsto p_I U_h$ *and* $e_1 \mapsto \beta_I w_3 U_h$ *for some* $\beta_I \in H^*\mathrm{BSpin}^h$ *satisfying* $\mathrm{Sq}^2\beta_I = p_I$ *(when* $I \in \mathcal{P}_{\text{odd}}$*).*

Proof. Taking the product of κ^I and ε^I over all partitions gives us a map

$$\mathrm{MSpin}^h \to \prod_{I \in \mathcal{P}_{\text{even}}} \mathrm{ksp}\langle 4|I|\rangle \times \prod_{I \in \mathcal{P}_{\text{odd}}} \Sigma^{4|I|} F.$$

Since there are only finitely many factors of this product with non-zero homotopy groups in a given degree, the map

$$\bigvee_{I \in \mathcal{P}_{\text{even}}} \mathrm{ksp}\langle 4|I|\rangle \vee \bigvee_{I \in \mathcal{P}_{\text{odd}}} \Sigma^{4|I|} F \to \prod_{I \in \mathcal{P}_{\text{even}}} \mathrm{ksp}\langle 4|I|\rangle \times \prod_{I \in \mathcal{P}_{\text{odd}}} \Sigma^{4|I|} F$$

is an equivalence. This induces the desired map

$$\mathrm{MSpin}^h \to \bigvee_{I \in \mathcal{P}_{\text{even}}} \mathrm{ksp}\langle 4|I|\rangle \vee \bigvee_{I \in \mathcal{P}_{\text{odd}}} \Sigma^{4|I|} F.$$

The effect of this map on cohomology follows from Proposition 5.21. □

6. Margolis homology of $H^*\mathrm{MSpin}^h$

In the preceding section, we constructed a map

$$\mathrm{MSpin}^h \to \bigvee_{I \in \mathcal{P}_{\text{even}}} \mathrm{ksp}\langle 4|I|\rangle \vee \bigvee_{I \in \mathcal{P}_{\text{odd}}} \Sigma^{4|I|} F.$$

The induced map on mod 2 cohomology takes the form

$$\bar{\theta} : \bigoplus_{I \in \mathcal{P}_{\text{even}}} \Sigma^{4|I|}(\mathcal{A} \otimes_{\mathcal{A}_1} \wr) \oplus \bigoplus_{I \in \mathcal{P}_{\text{odd}}} \Sigma^{4|I|}(\mathcal{A} \otimes_{\mathcal{A}_1} E) \to H^*\mathrm{MSpin}^h, \tag{6.1}$$

where $¿$ and E denote the upside-down question mark (Definition 4.13 and Corollary 4.8) and the elephant (Definition 4.12 and Lemma 5.4), respectively. Denote by $\bar{\theta}$ the homomorphism in Equation 6.1.

In this section, we will show that $\bar{\theta}$ induces isomorphisms in Margolis homology, analogous to a method used in [ABP67]. This will be used as a key input in Section 7, where we will show that $\bar{\theta}$ is injective and can be augmented to an isomorphism (which induces a map of spectra realizing the desired 2-local splitting).

Notation 6.1. If B is an $\mathcal{A}_1$-module, we will denote the $\mathcal{A}$-module $B_{\mathcal{A}} := \mathcal{A} \otimes_{\mathcal{A}_1} B$. Since $\mathcal{A}$ is free as a right $\mathcal{A}_1$-module, the functor $B \mapsto B_{\mathcal{A}}$ is exact. It follows that there is automatically an injective map of $\mathcal{A}_1$-modules $B \to B_{\mathcal{A}}$ given by $b \mapsto 1 \otimes b$.

Notation 6.2. Because mod 2 cohomology of MSpin, MSpin^c, and MSpin^h will show up so frequently later in this section, we will use the notation

$$\begin{aligned} M &:= H^*\mathrm{MSpin}, \\ M_c &:= H^*\mathrm{MSpin}^c, \\ M_h &:= H^*\mathrm{MSpin}^h. \end{aligned}$$

We will also write

$$\overline{N} := \bigoplus_{I \in \mathcal{P}_{\mathrm{even}}} \Sigma^{4|I|}\, ¿_{\mathcal{A}} \oplus \bigoplus_{I \in \mathcal{P}_{\mathrm{odd}}} \Sigma^{4|I|} E_{\mathcal{A}},$$

so that Equation 6.1 can be written as $\bar{\theta} : \overline{N} \to M_h$.

Setup 6.3. Let $Q_0 = \mathrm{Sq}^1$ and $Q_1 = \mathrm{Sq}^3 + \mathrm{Sq}^2\mathrm{Sq}^1$. These satisfy $Q_0^2 = Q_1^2 = 0$, so we can view multiplication by Q_0 or Q_1 as a differential of a chain complex on any $\mathcal{A}_1$-module (and by extension, any $\mathcal{A}$-module). Also, Q_0 and Q_1 are primitive elements of $\mathcal{A}$, so they act on products xy by $Q_i(xy) = (Q_i x)y + x(Q_i y)$.

Any map of $\mathcal{A}_1$-modules becomes a map of chain complexes with respect to the differentials $Q_0 \cdot (-)$ and $Q_1 \cdot (-)$. Given an $\mathcal{A}_1$-module B, we will denote by $H_*(B; Q_i)$ the homology of B with respect to Q_i. The usual results of homological algebra apply for computing $H_*(-; Q_i)$. In particular, short exact sequences of $\mathcal{A}_1$-modules induce long exact sequences in homology, and there is a Künneth theorem for $H_*(-; Q_i)$ [Mar83, Chapter 18.1, Propositions 1c and 2a].[5]

6.1. **The upside-down question mark and the elephant.**

We will begin by recalling a few basic computations of Q_i-homology, which we will then use to compute the Q_i-homology of $¿_{\mathcal{A}}$ and $E_{\mathcal{A}}$. To do so, we need to introduce a little more notation.

Notation 6.4. Let $\chi : \mathcal{A} \to \mathcal{A}$ denote the antipode of the Hopf algebra $\mathcal{A}$. We will frequently use the following properties of χ:

(i) $\chi(ab) = \chi(b)\chi(a)$ for all $a, b \in \mathcal{A}$.

[5] In Margolis's notation, we have $Q_0 = P_1^0$ and $Q_1 = P_1^0 P_2^1 + P_2^1 P_1^0$.

(ii) $\chi(Q_i) = Q_i$ for all i.

Let $\Delta_i = (a_1, a_2, a_3, \ldots)$, where $a_i = 1$ and $a_j = 0$ for $j \neq i$. Given a sequence R of natural numbers with finitely many non-zero terms, let $\mathrm{Sq}^R \in \mathcal{A}$ denote the Milnor basis vector associated to R. Finally, given a set V of vectors in some vector space, let $\langle V \rangle$ denote the span of V.

Lemma 6.5. *We have*

$$\begin{aligned}
H_*(\mathcal{A}/(\mathcal{A}\mathrm{Sq}^1 + \mathcal{A}\mathrm{Sq}^2); Q_0) &= \left\langle \chi(\mathrm{Sq}^{4k}) \;\middle|\; k \in \mathbb{N} \right\rangle, \\
H_*(\mathcal{A}/(\mathcal{A}\mathrm{Sq}^1 + \mathcal{A}\mathrm{Sq}^2); Q_1) &= \left\langle \chi(\mathrm{Sq}^{2\sum_{j=1}^k \Delta_{i_j}}) \;\middle|\; k \in \mathbb{N},\ i_1 > \ldots > i_k \geq 2 \right\rangle, \\
H_*(\mathcal{A}/\mathcal{A}\mathrm{Sq}^3; Q_0) &= \left\langle \chi(\mathrm{Sq}^{4k})\mathrm{Sq}^2 \;\middle|\; k \in \mathbb{N} \right\rangle, \\
H_*(\mathcal{A}/\mathcal{A}\mathrm{Sq}^3; Q_1) &= \left\langle \chi(\mathrm{Sq}^{2\sum_{j=1}^k \Delta_{i_j}})\mathrm{Sq}^2 \;\middle|\; k \in \mathbb{N},\ i_1 > \ldots > i_k \geq 2 \right\rangle.
\end{aligned}$$

Proof. This is [ABP67, Theorem 6.9]. $\square$

The following is a sort of Leibniz rule for the Q_i-differentials.

Lemma 6.6. *For any natural number n and any distinct natural numbers $i_1, \ldots, i_k$, we have*

$$\begin{aligned}
Q_1\mathrm{Sq}^{2n-2} &= Q_0\mathrm{Sq}^{2n} + \mathrm{Sq}^{2n}Q_0, \\
0 &= Q_1\mathrm{Sq}^{2\sum_{j=1}^k \Delta_{i_j}} + \mathrm{Sq}^{2\sum_{j=1}^k \Delta_{i_j}}Q_1.
\end{aligned}$$

Proof. In the Milnor basis, we have

$$Q_i\mathrm{Sq}^{2I} + \mathrm{Sq}^{2I}Q_i = \sum_{j=1}^{\infty} Q_{i+j}\mathrm{Sq}^{2(I-2^i\Delta_j)}$$

for any partition I [Mil58, Theorem 4a]. We also have $\mathrm{Sq}^n = \mathrm{Sq}^{n\Delta_1}$ (see [Mil58, Section 6]) and $\mathrm{Sq}^R = 0$ if any term of R is negative [Mil58, p. 163]. Setting $i = 0$ and $I = (n)$, we thus compute

$$\begin{aligned}
Q_0\mathrm{Sq}^{2n} + \mathrm{Sq}^{2n}Q_0 &= \sum_{j=1}^{\infty} Q_j\mathrm{Sq}^{2(n\Delta_1 - \Delta_j)} \\
&= Q_1\mathrm{Sq}^{2n-2} + \sum_{j=2}^{\infty} Q_j \cdot 0.
\end{aligned}$$

Setting $i = 1$ and $I = \sum_{j=1}^k \Delta_{i_j}$, we compute

$$\begin{aligned}
Q_1\mathrm{Sq}^{2\sum_{j=1}^k \Delta_{i_j}} + \mathrm{Sq}^{2\sum_{j=1}^k \Delta_{i_j}}Q_1 &= \sum_{\ell=1}^{\infty} Q_{\ell+1}\mathrm{Sq}^{2(\sum_{j=1}^k \Delta_{i_j} - 2\Delta_\ell)} \\
&= \sum_{\ell=1}^{\infty} Q_{\ell+1} \cdot 0,
\end{aligned}$$

since $\sum_{j=1}^{k} \Delta_{i_j} - 2\Delta_\ell$ always contains a negative term. $\square$

Using Lemmas 6.5 and 6.6, we are able to compute the homologies of both $\wr_{\mathcal{A}}$ and $E_{\mathcal{A}}$. In order to simplify the presentation of $H_*(\wr_{\mathcal{A}}; Q_i)$ and $H_*(E_{\mathcal{A}}; Q_i)$, we need another lemma.

Lemma 6.7. *We have* $\chi(\mathrm{Sq}^{4k-2})\mathrm{Sq}^2 \in \mathcal{A}\mathrm{Sq}^1$ *for any* $k \in \mathbb{N}$.

Proof. If $k = 0$, then $\chi(\mathrm{Sq}^{4k-2}) = 0$. If $k = 1$, then

$$\begin{aligned} \chi(\mathrm{Sq}^{4k-2})\mathrm{Sq}^2 &= \chi(\mathrm{Sq}^2)\mathrm{Sq}^2 \\ &= (\mathrm{Sq}^2)^2 \\ &= \mathrm{Sq}^3\mathrm{Sq}^1 \in \mathcal{A}\mathrm{Sq}^1. \end{aligned}$$

Finally, suppose $k \geq 2$. We have the Adem relations

$$\mathrm{Sq}^i\mathrm{Sq}^j = \sum_{t=0}^{\lfloor \frac{i}{2} \rfloor} \binom{j-t-1}{i-2t} \mathrm{Sq}^{i+j-t}\mathrm{Sq}^t$$

when $i < 2j$. We then compute

$$\begin{aligned} \mathrm{Sq}^2\mathrm{Sq}^n &= \sum_{t=0}^{1} \binom{n-t-1}{2-2t} \mathrm{Sq}^{n+2-t}\mathrm{Sq}^t \\ &= \binom{n-1}{2}\mathrm{Sq}^{n+2} + \mathrm{Sq}^{n+1}\mathrm{Sq}^1 \\ &= \begin{cases} \mathrm{Sq}^{n+2} + \mathrm{Sq}^{n+1}\mathrm{Sq}^1 & n = 0, 3 \pmod 4, \\ \mathrm{Sq}^{n+1}\mathrm{Sq}^1 & n = 1, 2 \pmod 4 \end{cases} \end{aligned}$$

if $2 < 2n$. This implies that

$$\begin{aligned} \mathrm{Sq}^2\mathrm{Sq}^{4k-4} + \mathrm{Sq}^2\mathrm{Sq}^{4k-5}\mathrm{Sq}^1 &= \mathrm{Sq}^{4k-2} + \mathrm{Sq}^{4k-3}\mathrm{Sq}^1 + (\mathrm{Sq}^{4k-3} + \mathrm{Sq}^{4k-4}\mathrm{Sq}^1)\mathrm{Sq}^1 \\ &= \mathrm{Sq}^{4k-2}. \end{aligned}$$

Therefore

$$\begin{aligned} \chi(\mathrm{Sq}^{4k-2})\mathrm{Sq}^2 &= \chi(\mathrm{Sq}^2(\mathrm{Sq}^{4k-4} + \mathrm{Sq}^{4k-5}\mathrm{Sq}^1))\mathrm{Sq}^2 \\ &= \chi(\mathrm{Sq}^{4k-4} + \mathrm{Sq}^{4k-5}\mathrm{Sq}^1)\chi(\mathrm{Sq}^2)\mathrm{Sq}^2 \\ &= \chi(\mathrm{Sq}^{4k-4} + \mathrm{Sq}^{4k-5}\mathrm{Sq}^1)\mathrm{Sq}^2\mathrm{Sq}^2 \\ &= \chi(\mathrm{Sq}^{4k-4} + \mathrm{Sq}^{4k-5}\mathrm{Sq}^1)\mathrm{Sq}^3\mathrm{Sq}^1. \end{aligned}$$

$\square$

Proposition 6.8. *The Margolis homology of* $\wr_{\mathcal{A}}$ *has the following presentation:*

$$\begin{aligned} H_*(\wr_{\mathcal{A}}; Q_0) &\cong \left\langle \chi(\mathrm{Sq}^{4k})q_0 \;\middle|\; k \in \mathbb{N} \right\rangle, \\ H_*(\wr_{\mathcal{A}}; Q_1) &\cong \left\langle \chi(\mathrm{Sq}^{2\sum_{j=1}^{k} \Delta_{i_j}})\mathrm{Sq}^2 q_0 \;\middle|\; k \in \mathbb{N},\ i_1 > \ldots > i_k \geq 2 \right\rangle. \end{aligned}$$

Proof. There is a short exact sequence

$$0 \longrightarrow \mathcal{A}_1\mathrm{Sq}^3 \longrightarrow \mathcal{A}_1 \longrightarrow \mathcal{A}_1/\mathcal{A}_1\mathrm{Sq}^3 \longrightarrow 0,$$

and an isomorphism $\mathcal{A}_1\mathrm{Sq}^3 \cong \Sigma^3\text{¿}$, where Sq^3 corresponds to q_0. So tensoring with $\mathcal{A}$ gives us a short exact sequence

$$0 \longrightarrow \Sigma^3\text{¿}_{\mathcal{A}} \longrightarrow \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{A}\mathrm{Sq}^3 \longrightarrow 0,$$

which induces a long exact sequence in homology:

$$H_*(\mathcal{A};Q_i) \longrightarrow H_*(\mathcal{A}/\mathcal{A}\mathrm{Sq}^3;Q_i) \longrightarrow H_{*+\deg Q_i}(\Sigma^3\text{¿}_{\mathcal{A}};Q_i) \longrightarrow H_{*+\deg Q_i}(\mathcal{A};Q_i).$$

The homologies of $\mathcal{A}$ vanish [Mar83, p. 331, Proposition 1], so the connecting homomorphism

$$H_*(\mathcal{A}/\mathcal{A}\mathrm{Sq}^3;Q_i) \to H_{*+\deg Q_i}(\Sigma^3\text{¿}_{\mathcal{A}};Q_i) \tag{6.2}$$

is an isomorphism. Since the homology of $\Sigma^3\text{¿}_{\mathcal{A}}$ is a degree 3 shift of the homology of $\text{¿}_{\mathcal{A}}$, there is an isomorphism $H_*(\mathcal{A}/\mathcal{A}\mathrm{Sq}^3;Q_i) \to H_{*+\deg Q_i-3}(\text{¿}_{\mathcal{A}};Q_i)$.

A presentation of $H_*(\mathcal{A}/\mathcal{A}\mathrm{Sq}^3;Q_i)$ is given in Lemma 6.5. All that remains is to give a formula for the connecting homomorphism (Equation 6.2). The connecting homomorphism can be computed for a class in $H_*(\mathcal{A}/\mathcal{A}\mathrm{Sq}^3;Q_i)$ by choosing a representative cycle in $\mathcal{A}/\mathcal{A}\mathrm{Sq}^3$, lifting it to an element of $\mathcal{A}$, acting by Q_i to get a cycle in $\text{¿}_{\mathcal{A}}$, and taking the resulting homology class.

For Q_0, we lift $\chi(\mathrm{Sq}^{4k})\mathrm{Sq}^2$ and compute

$$\begin{aligned}
Q_0\chi(\mathrm{Sq}^{4k})\mathrm{Sq}^2 &= \chi(Q_0)\chi(\mathrm{Sq}^{4k})\mathrm{Sq}^2 \\
&= \chi(\mathrm{Sq}^{4k}Q_0)\mathrm{Sq}^2 \\
\text{(Lemma 6.6)} \qquad &= \chi(Q_0\mathrm{Sq}^{4k} + Q_1\mathrm{Sq}^{4k-2})\mathrm{Sq}^2 \\
&= \chi(\mathrm{Sq}^{4k})Q_0\mathrm{Sq}^2 + \chi(\mathrm{Sq}^{4k-2})Q_1\mathrm{Sq}^2 \\
\text{(Setup 6.3)} \qquad &= \chi(\mathrm{Sq}^{4k})\mathrm{Sq}^3 + \chi(\mathrm{Sq}^{4k-2})\mathrm{Sq}^2\mathrm{Sq}^3.
\end{aligned}$$

Lemma 6.7 implies that $\chi(\mathrm{Sq}^{4k-2})\mathrm{Sq}^2 \in \mathcal{A}\mathrm{Sq}^1$, so the second term vanishes (because $\mathrm{Sq}^1\mathrm{Sq}^3 = 0$). Finally, the isomorphism $\mathcal{A}\mathrm{Sq}^3 \cong \Sigma^3\text{¿}_{\mathcal{A}}$ has Sq^3 in correspondence with q_0, so the Q_0-homology of $\text{¿}_{\mathcal{A}}$ is generated by $\chi(\mathrm{Sq}^{4k})q_0$.

For Q_1, we lift $\chi(\mathrm{Sq}^{2\sum_{j=1}^k \Delta_{i_j}})\mathrm{Sq}^2$ and compute

$$\begin{aligned}
Q_1\chi(\mathrm{Sq}^{2\sum_{j=1}^k \Delta_{i_j}})\mathrm{Sq}^2 &= \chi(\mathrm{Sq}^{2\sum_{j=1}^k \Delta_{i_j}}Q_1)\mathrm{Sq}^2 \\
\text{(Lemma 6.6)} \qquad &= \chi(Q_1\mathrm{Sq}^{2\sum_{j=1}^k \Delta_{i_j}})\mathrm{Sq}^2 \\
&= \chi(\mathrm{Sq}^{2\sum_{j=1}^k \Delta_{i_j}})Q_1\mathrm{Sq}^2 \\
\text{(Setup 6.3)} \qquad &= \chi(\mathrm{Sq}^{2\sum_{j=1}^k \Delta_{i_j}})\mathrm{Sq}^2\mathrm{Sq}^3.
\end{aligned}$$

Therefore the Q_1-homology of $\text{¿}_{\mathcal{A}}$ is generated by $\chi(\mathrm{Sq}^{2\sum_{j=1}^k \Delta_{i_j}})\mathrm{Sq}^2q_0$. $\square$

Proposition 6.9. *The Margolis homology of $E_{\mathcal{A}}$ has the following presentation:*

$$H_*(E_{\mathcal{A}};Q_0) \cong \left\langle \chi(\mathrm{Sq}^{4k})e_0 + \chi(\mathrm{Sq}^{4k-2})\mathrm{Sq}^1 e_1 \;\middle|\; k \in \mathbb{N} \right\rangle,$$
$$H_*(E_{\mathcal{A}};Q_1) \cong \left\langle \chi(\mathrm{Sq}^{2\sum_{j=1}^k \Delta_{i_j}})(\mathrm{Sq}^2 e_0 + \mathrm{Sq}^1 e_1) \;\middle|\; k \in \mathbb{N},\ i_1 > \ldots > i_k \geq 2 \right\rangle.$$

Proof. There is a short exact sequence

$$0 \longrightarrow \mathcal{A}_1\mathrm{Sq}^1 + \mathcal{A}_1\mathrm{Sq}^2 \longrightarrow \mathcal{A}_1 \longrightarrow \mathcal{A}_1/(\mathcal{A}_1\mathrm{Sq}^1 + \mathcal{A}_1\mathrm{Sq}^2) \longrightarrow 0$$

and an isomorphism $\mathcal{A}_1\mathrm{Sq}^1 + \mathcal{A}_1\mathrm{Sq}^2 \cong \Sigma E$, where Sq^1 corresponds to e_0 and Sq^2 corresponds to e_1. Tensoring with $\mathcal{A}$ thus gives us a short exact sequence

$$0 \longrightarrow \Sigma E_{\mathcal{A}} \longrightarrow \mathcal{A} \longrightarrow \mathcal{A}/(\mathcal{A}\mathrm{Sq}^1 + \mathcal{A}\mathrm{Sq}^2) \longrightarrow 0.$$

Since $H_*(\mathcal{A};Q_i) = 0$ [Mar83, p. 331, Proposition 1], we see that the connecting homomorphism in the induced long exact sequence on Q_i-homology is an isomorphism

$$H_*(\mathcal{A}/(\mathcal{A}\mathrm{Sq}^1 + \mathcal{A}\mathrm{Sq}^2);Q_i) \to H_{*+\deg Q_i - 1}(E_{\mathcal{A}};Q_i).$$

We have already calculated $H_*(\mathcal{A}/(\mathcal{A}\mathrm{Sq}^1+\mathcal{A}\mathrm{Sq}^2);Q_i)$ in Lemma 6.5, so we can compute the connecting homomorphism for Q_0-homology by

$$\begin{aligned}
& Q_0\chi(\mathrm{Sq}^{4k}) = \chi(\mathrm{Sq}^{4k}Q_0) \\
\text{(Lemma 6.6)}\quad & = \chi(Q_0\mathrm{Sq}^{4k} + Q_1\mathrm{Sq}^{4n-2}) \\
& = \chi(\mathrm{Sq}^{4k})Q_0 + \chi(\mathrm{Sq}^{4n-2})Q_1 \\
\text{(Setup 6.3)}\quad & = \chi(\mathrm{Sq}^{4k})\mathrm{Sq}^1 + \chi(\mathrm{Sq}^{4n-2})\mathrm{Sq}^2\mathrm{Sq}^1 + \chi(\mathrm{Sq}^{4n-2})\mathrm{Sq}^1\mathrm{Sq}^2.
\end{aligned}$$

Since $\chi(\mathrm{Sq}^{4n-2})\mathrm{Sq}^2 \in \mathcal{A}\mathrm{Sq}^1$ by Lemma 6.7, the term $\chi(\mathrm{Sq}^{4n-2})\mathrm{Sq}^2\mathrm{Sq}^1$ vanishes because $(\mathrm{Sq}^1)^2 = 0$. Since the isomorphism $\mathcal{A}\mathrm{Sq}^1+\mathcal{A}\mathrm{Sq}^2 \to \Sigma E_{\mathcal{A}}$ satisfies $\mathrm{Sq}^1 \mapsto e_0$ and $\mathrm{Sq}^2 \mapsto e_1$, we find that $H_*(E_{\mathcal{A}};Q_0)$ is generated by $\chi(\mathrm{Sq}^{4k})e_0 + \chi(\mathrm{Sq}^{4k-2})\mathrm{Sq}^1 e_1$.

For Q_1, we calculate

$$\begin{aligned}
& Q_1\chi(\mathrm{Sq}^{2\sum_{j=1}^k \Delta_{i_j}}) = \chi(\mathrm{Sq}^{2\sum_{j=1}^k \Delta_{i_j}}Q_1) \\
\text{(Lemma 6.6)}\quad & = \chi(Q_1\mathrm{Sq}^{2\sum_{j=1}^k \Delta_{i_j}}) \\
& = \chi(\mathrm{Sq}^{2\sum_{j=1}^k \Delta_{i_j}})Q_1 \\
\text{(Setup 6.3)}\quad & = \chi(\mathrm{Sq}^{2\sum_{j=1}^k \Delta_{i_j}})(\mathrm{Sq}^2\mathrm{Sq}^1 + \mathrm{Sq}^1\mathrm{Sq}^2),
\end{aligned}$$

so $H_*(E_{\mathcal{A}};Q_1)$ is generated by $\chi(\mathrm{Sq}^{2\sum_{j=1}^k \Delta_{i_j}})(\mathrm{Sq}^2 e_0 + \mathrm{Sq}^1 e_1)$. ☐

6.2. **Q_i-homology of $H^*\mathrm{MSpin}^h$.** Our next goal is to calculate the Q_i-homology of $M_h := H^*\mathrm{MSpin}^h$. To begin, recall the Wu formula:

$$\mathrm{Sq}^i w_j = \sum_{t=0}^{i} \binom{j+t-i-1}{t} w_{i-t}w_{j+t}. \tag{6.3}$$

Evaluating the action of Sq^1 and using the fact that $w_1 = 0$ in $H^*\mathrm{BSpin}^h$, we can compute $Q_0 w_j$ and $Q_1 w_j$.

Lemma 6.10. *For any $j \in \mathbb{N}$,*

$$Q_0 w_j = \begin{cases} w_{j+1} & j \text{ is even,} \\ 0 & j \text{ is odd,} \end{cases} \qquad \text{and} \qquad Q_1 w_j = \begin{cases} w_{j+3} + w_3 w_j & j \text{ is even,} \\ w_3 w_j & j \text{ is odd} \end{cases}$$

in $H^*\mathrm{BSpin}^h$.

Proof. Using Equation 6.3, we find

$$\begin{aligned} Q_0 w_j &= \binom{j-2}{0} w_1 w_j + \binom{j-1}{1} w_0 w_{j+1} \\ &= (j-1) w_{j+1}. \end{aligned}$$

For Q_1, we need to evaluate Sq^2 and Sq^3 as well:

$$\begin{aligned} \mathrm{Sq}^2 w_j &= w_2 w_j + \binom{j-1}{2} w_{j+2}, \\ \mathrm{Sq}^3 w_j &= w_3 w_j + (j-3) w_2 w_{j+2} + \binom{j-1}{3} w_{j+3}. \end{aligned}$$

Putting these together gives us

$$\begin{aligned} Q_1 w_j &= (\mathrm{Sq}^3 + \mathrm{Sq}^2\mathrm{Sq}^1) w_j \\ &= w_3 w_j + (j-3) w_2 w_{j+2} + \binom{j-1}{3} w_{j+3} + (j-1)\mathrm{Sq}^2 w_{j+1} \\ &= w_3 w_j + (j-3) w_2 w_{j+2} + \binom{j-1}{3} w_{j+3} + (j-1) w_2 w_{j+1} + (j-1)\binom{j}{2} w_{j+3} \\ &= w_3 w_j + \left(\binom{j-1}{3} + (j-1)\binom{j}{2} \right) w_{j+3}. \end{aligned}$$

It remains to determine the parity of $\binom{j-1}{3} + (j-1)\binom{j}{2}$. Note $\binom{j}{2} = \frac{1}{2} j(j-1)$ is even if and only if $j = 0, 1 \pmod 4$, and $\binom{j-1}{3} = \frac{1}{6}(j-1)(j-2)(j-3)$ is even if and only if $j = 1, 2, 3 \pmod 4$. Thus $\binom{j-1}{3} + (j-1)\binom{j}{2}$ is odd if j is even and is even if j is odd. $\square$

As an application of Lemma 6.10, we prove the following lemma used in the proof of Proposition 4.5.

Lemma 6.11. *The class $w_{2i_1}^2 \cdots w_{2i_s}^2 \in \mathrm{BSpin}^h$ is not in the image of Sq^1.*

Proof. Let $B = H^*\mathrm{BSpin}^h$. By [Hu22, Corollary 2.35],

$$H_*(B; Q_0) = \mathbb{Z}/2\mathbb{Z}[w_2^2, w_{2k}^2, \nu_{2^r} \mid k \neq 2^r, r > 1],$$

where ν_i is the ith Wu class. For our purposes, the only fact about the Wu classes we need is that ν_{2^r} is w_{2^r} plus products of lower degree classes [Sto68, p. 316]. In particular, we can write

$$\nu_{2^r} = w_{2^r} + q_r(w_2, \ldots, w_{2^r-2}) + \sum_{j=1}^{2^r-1} w_j x_{r,j}$$

for some polynomial q_r and some $x_{r,j}$. Since we are working over $\mathbb{Z}/2\mathbb{Z}$, the freshman's dream gives us

$$\begin{aligned} w_{2^r}^2 &= \nu_{2^r}^2 + q_r(w_2^2, \ldots, w_{2^r-2}^2) + \sum_{j=1}^{2^r-1} w_j^2 x_{r,j}^2 \\ &= \nu_{2^r}^2 + q_r(w_2^2, \ldots, w_{2^r-2}^2) + \sum_{j=1}^{2^r-1} Q_0(w_{j-1} w_j x_{r,j}^2). \end{aligned} \tag{6.4}$$

Expanding out $w = w_{2i_1}^2 \cdots w_{2i_s}^2$ (using Equation 6.4 if necessary), we see that w cannot be in the image of Sq^1. Indeed, the expansion of w is a sum of monomials in $R := \mathbb{Z}/2\mathbb{Z}[w_2^2, w_{2k}^2, \nu_{2^r}]$ and products of monomials of R and terms of the form

$$Q_0(w_{j-1} w_j x_{r,j}^2). \tag{6.5}$$

Modulo terms of the form in Equation 6.5 (which lie in the image of $Q_0 = \mathrm{Sq}^1$), the class w is a non-zero sum of linearly independent monomials that generate $H_*(B; Q_0)$. As generators of $_*(B; Q_0)$, such monomials do not lie in the image of $Q_0 \cdot -$, so $w \notin \mathrm{im}(Q_0 \cdot -)$. □

To calculate $H_*(M_h; Q_i)$, we will use methods similar to those of [ABP67].[6] The difficult part of this computation is managing the Stiefel–Whitney classes in $H^*\mathrm{BSpin}^h$ that hit a decomposable Stiefel–Whitney class after applying the Q_i-differential. The resolution is that there is always a way to replace these classes. To construct our replacement generators, we need another lemma.

Lemma 6.12. *The map*

$$\begin{aligned} \mathcal{A} &\to M_h \\ 1 &\mapsto U_h \end{aligned}$$

factors through $¿_{\mathcal{A}} \cong \mathcal{A}/(\mathcal{A}\mathrm{Sq}^1 + \mathcal{A}(\mathrm{Sq}^5 + \mathrm{Sq}^4\mathrm{Sq}^1))$.

Proof. This is true because $\mathrm{Sq}^1 U_h = w_1 U_h = 0$ and

$$\begin{aligned} (\mathrm{Sq}^5 + \mathrm{Sq}^4\mathrm{Sq}^1)U_h &= \mathrm{Sq}^2\mathrm{Sq}^3 U_h \\ &= \mathrm{Sq}^2(w_3 U_h) \\ &= (\mathrm{Sq}^2 w_3)U_h + (\mathrm{Sq}^1 w_3)(\mathrm{Sq}^1 U_h) + w_3(\mathrm{Sq}^2 U_h) \\ \text{(Equation 6.3)} \quad &= (w_2 w_3 + w_5)U_h + w_3 w_2 U_h \\ (w_5 \in H^5\mathrm{BSpin}^h \text{ vanishes}) \quad &= w_2 w_3 U_h + w_2 w_3 U_h \\ &= 0. \end{aligned}$$

□

Corollary 6.13. *Each cycle in* $H_*(¿_{\mathcal{A}}; Q_i)$ *maps to a cycle in* $H_*(M_h; Q_i)$.

Proof. This follows by applying $H_*(-; Q_i)$ to the factorization $\mathcal{A} \to ¿_{\mathcal{A}} \to M_h$. □

[6] One could alternatively use the shearing map (Lemma 2.7) for this calculation.

The final ingredient we need before computing $H_*(M_h; Q_0)$ is an alternative presentation of $H^*\mathrm{BSpin}^h$.

Lemma 6.14. *For each $k \geq 2$, there are cohomology classes $f_{2^k} \in H^{2^k}\mathrm{BSpin}^h$ such that $Q_0 f_{2^k} = 0$ and*

$$H^*\mathrm{BSpin}^h \cong \mathbb{Z}/2\mathbb{Z}\left[w_i, f_{2^k} \mid k \geq 2, i \geq 2, i \neq 2^r \text{ or } 2^r+1 \text{ for } r \geq 2\right].$$

Proof. There are classes $f_{2^k} \in H^{2^k}\mathrm{BSpin}^h$ such that $\chi(\mathrm{Sq}^{2^k})U_h = \chi(\mathrm{Sq}^{4\cdot 2^{k-2}})U_h = f_{2^k}U_h$, as the Thom isomorphism theorem implies every element of $H^*\mathrm{MSpin}^h$ can be written in the form fU_h for some $f \in H^*\mathrm{BSpin}^h$. Since $Q_0U_h = 0$ by Lemma 6.10, we have

$$\begin{aligned} 0 &= Q_0(f_{2^k}U_h) \\ &= (Q_0 f_{2^k})U_h + f_{2^k}(Q_0U_h) \\ &= (Q_0 f_{2^k})U_h. \end{aligned}$$

This implies $Q_0 f_{2^k} = 0$. From the usual presentation of $H^*\mathrm{BSpin}^h$ (Proposition 4.3), it remains to show that $f_{2^k} \equiv w_{2^k} \mod (w_1, \ldots, w_{2^k-1})$. To prove this, write $\chi(\mathrm{Sq}^{2^k}) = \mathrm{Sq}^{2^k} + a$ for some $a \in \mathcal{A}$, so that

$$\begin{aligned} f_{2^k}U_h &= (\mathrm{Sq}^{2^k} + a)U_h \\ &= w_{2^k}U_h + aU_h. \end{aligned}$$

If we write aU_h in the monomial basis, we want to show that the coefficient of $w_{2^k}U_h$ is zero. Since $\chi(\mathrm{Sq}^{2^k}) \equiv \mathrm{Sq}^{2^k} \mod (\mathrm{Sq}^1, \ldots, \mathrm{Sq}^{2^k-1})$ [Mil58, Section 7], we know that $a \in (\mathrm{Sq}^1, \ldots, \mathrm{Sq}^{2^k-1})$. Now expand out the terms of aU_h by using the action of the Steenrod squares on U_h and the Wu formula. None of the resulting monomials can have degree equal to that of w_{2^k}, so $a \in (w_1, \ldots, w_{2^k-1})$. □

We are now set to compute $H_*(M_h; Q_0)$.

Lemma 6.15. *Let $f_{2^k} \in H^{2^k}\mathrm{BSpin}^h$ be as in Lemma 6.14. Let*

$$R := \mathbb{Z}/2\mathbb{Z}\left[w_{2i}^2, f_{2^k} \;\middle|\; k \geq 2, i \geq 3, i \neq 2^{r-1} \text{ for } r \geq 2\right].$$

Then $H_*(M_h; Q_0)$ is the free R-module generated by $U_h \in M_h$.

Proof. We first use the Künneth theorem to break up the calculation into manageable pieces. By Lemma 6.14, we can decompose $H^*\mathrm{BSpin}^h$ the following tensor product:

$$H^*\mathrm{BSpin}^h \cong \bigotimes_{k\geq 2} \mathbb{Z}/2\mathbb{Z}[f_{2^k}] \otimes \bigotimes_{i\neq 2^{r-1}, r\geq 2} \mathbb{Z}/2\mathbb{Z}[w_{2i}, w_{2i+1}].$$

Each factor is well-defined as a module over the exterior algebra generated by Q_0, since $Q_0 f_{2^k} = Q_0 w_{2i+1} = 0$ and $Q_0 w_{2i} = w_{2i+1}$. Every monomial in $\mathbb{Z}/2\mathbb{Z}[f_{2^k}]$ is a cycle, so $H_*(\mathbb{Z}/2\mathbb{Z}[f_{2^k}]; Q_0) \cong \mathbb{Z}/2\mathbb{Z}\,[f_{2^k}]$. For the other factors, we have

$$Q_0(w_{2i}^a w_{2i+1}^b) = a w_{2i}^{a-1} w_{2i+1}^{b+1}.$$

It follows that $\ker(Q_0 \cdot -)$ is the subspace generated by those monomials having an even number of w_{2i} factors, and that $\operatorname{im}(Q_0 \cdot -)$ is the subspace generated by those monomials having an even number of w_{2i} factors and at least one w_{2i+1}. Hence the homology is generated by monomials having an even number of w_{2i} factors and no factors of w_{2i+1}, so $H_*(\mathbb{Z}/2\mathbb{Z}\,[w_{2i}, w_{2i+1}]\,; Q_0) \cong \mathbb{Z}/2\mathbb{Z}\,[w_{2i}^2]$. Using the Künneth theorem over a field, we see that

$$H_*(H^*\mathrm{BSpin}^h; Q_0) \cong R.$$

Since the Thom class U_h satisfies $Q_0 U_h = 0$, the Thom isomorphism $x \mapsto xU_h$ is a map of chain complexes. It follows that $H_*(M_h; Q_0) \cong R \cdot U_h$, as desired. $\square$

To compute $H_*(M_h; Q_1)$, we again need an alternative presentation of $H^*\mathrm{BSpin}^h$.

Lemma 6.16. *There are classes $t_{2j+1} \in H^{2j+1}\mathrm{BSpin}^h$ for $j \geq 3$ and $j \neq 2^m$, and classes $g_{2^k-2} \in H^{2^k-2}\mathrm{BSpin}^h$ for $k \geq 3$, such that $Q_1 t_{2j+1} = Q_1 g_{2^k-2} = 0$ and*

$$H^*\mathrm{BSpin}^h \cong \mathbb{Z}/2\mathbb{Z}\left[w_3, w_{2i}, t_{2j+1}, g_{2^r-2} \;\middle|\; i \neq 2^{r-1}-1, r \geq 3, j \geq 3, j \neq 2^m\right].$$

Proof. Define $t_{2j+1} := w_{2j+1} + w_3 w_{2j-2}$. Since $j \neq 2^m$ for $m \geq 1$, we have $2j+1 \neq 2^{m+1}+1$ and hence w_{2j+1} is one of the polynomial generators of $H^*\mathrm{BSpin}^h$. It follows from Proposition 4.3 that

$$H^*\mathrm{BSpin}^h \cong \mathbb{Z}/2\mathbb{Z}\left[w_{2i}, w_3, t_{2j+1} \;\middle|\; j \neq 2^{r-1}\right].$$

To see that $Q_1 t_{2j+1} = 0$, note that $Q_1 w_{2j-2} = w_{2j+1} + w_3 w_{2j-2} = t_{2j+1}$ and recall that $Q_1^2 = 0$.

Next, we employ similar tactics as in Lemma 6.14 to construct the classes g_{2^k-2}, although slightly more work is needed due to the fact that U_h is not a Q_1-cycle. To define the g_{2^r-2}, we induct on $r \geq 3$, using the same argument for the base case and inductive step. Specifically, we assume that the g_{2^q-2} have been constructed for $q < r$, that $Q_1 g_{2^q-2} = 0$, and that

$$H^*\mathrm{BSpin}^h \cong \mathbb{Z}/2\mathbb{Z}\left[w_3, w_{2i}, t_{2j+1}, g_{2^q-2} \;\middle|\; i \neq 2^{q-1}-1, 3 \leq q < r, j \geq 3, j \neq 2^m\right]. \tag{6.6}$$

By our computation of $H_*(\text{¿}_{\mathcal{A}}; Q_1)$ (Proposition 6.8), we see that $\chi(\mathrm{Sq}^{2\Delta_{r-1}})\mathrm{Sq}^2 U_h$ is a Q_1-cycle. To extract a replacement of w_{2^r-2} from this, first write $\chi(\mathrm{Sq}^{2\Delta_{r-1}})\mathrm{Sq}^2 U_h = aw_2 U_h + bU_h$, where aw_2 and b are classes of degree 2^r and no monomials of b (in the basis given by Equation 6.6) are multiples of w_2. We will set $a := g_{2^r-2}$.

We first need to verify that $Q_1 g_{2^r-2} = 0$. To this end, we compute

$$\begin{aligned}
&(Q_1\text{-cycle}) & 0 &= Q_1(\chi(\mathrm{Sq}^{2\Delta_{r-1}})\mathrm{Sq}^2 U_h)\\
& & &= Q_1(aw_2 U_h + bU_h)\\
&(Q_1 = \mathrm{Sq}^3 + \mathrm{Sq}^2\mathrm{Sq}^1) & &= (Q_1 a)w_2 U_h + (Q_1 b)U_h + bw_3 U_h.
\end{aligned}$$

Note that in the monomial basis, no term of $(Q_1 b)U_h$ are divisible by w_2. Indeed, no terms of b are divisible by w_2, and the images under Q_1 of the basis elements are

$$\begin{aligned} Q_1 w_{2i} &= t_{2i+3} = w_{2i+3} + w_3 w_{2i}, \\ Q_1 w_3 &= w_3^2, \\ Qt_{2j+1} &= 0, \\ Qg_{2^q-2} &= 0, \end{aligned}$$

none of which are divisible by w_2. It follows that no terms of bw_3U_h are divisible by w_2, so $(Q_1 a)w_2U_h = 0$ and therefore $Q_1 a = 0$.

It remains to show that a is w_{2^r-2} plus products of lower generators. We will revert to the Stiefel–Whitney generators of $H^*\mathrm{BSpin}^h$ for this step, since rewriting the Stiefel–Whitney generators in terms of the new generators will not introduce monomials with w_{2^r-2} for degree reasons. By [ABP67, Proposition 6.2], $\chi(\mathrm{Sq}^{2\Delta_{r-1}})$ is Sq^{2^r-2} modulo admissible sequences with two or more factors. By expanding out the action of these other admissible sequences using the Wu formula, we find that the monomial w_{2^r-2} cannot arise from these admissible sequences. It follows that a is indeed w_{2^r-2} modulo products of lower degree terms. □

Now we compute $H_*(M_h; Q_1)$.

Lemma 6.17. *Let $g_{2^k-2} \in H^{2^k-2}\mathrm{BSpin}^h$ be as in Lemma 6.16. Let*

$$S := \mathbb{Z}/2\mathbb{Z}\left[w_{2i}^2, g_{2^r-2} \;\middle|\; i \neq 2^{r-1}-1, r \geq 3\right] w_2 U.$$

Then $H_*(M_h; Q_1)$ is the free S-module generated by w_2U_h.

Proof. By Lemma 6.16, the Thom isomorphism, and the Künneth formula, M_h can be written as the tensor product

$$\mathbb{Z}/2\mathbb{Z}[w_2, w_3]U_h \otimes \bigotimes_{\substack{i\geq 2 \\ i\neq 2^m-1}} \mathbb{Z}/2\mathbb{Z}[w_{2i}, t_{2i+3}] \otimes \bigotimes_{r\geq 3} \mathbb{Z}/2\mathbb{Z}[g_{2^r-2}].$$

Moreover, each of these factors is closed under the action of Q_1 because

$$\begin{aligned} Q_1 U_h &= w_3 U_h, & Q_1 w_{2i} &= t_{2i+3}, \\ Q_1 w_2 &= w_2 w_3, & Q_1 t_{2i+3} &= 0, \\ Q_1 w_3 &= w_3^2, & Q_1 g_{2^r-2} &= 0. \end{aligned} \tag{6.7}$$

In order to determine $H_*(M_h; Q_1)$, it thus suffices to compute the Q_1-homology of each factor individually. For $\mathbb{Z}/2\mathbb{Z}[w_2, w_3]U_h$, the action of Q_1 on the monomial $w_2^a w_3^b U_h$ is

$$\begin{aligned} Q_1(w_2^a w_3^b U_h) &= a w_2^a w_3^{b+1} U_h + b w_2^a w_3^{b+1} U_h + w_2^a w_3^{b+1} U_h \\ &= (a+b+1) w_2^a w_3^{b+1} U_h. \end{aligned}$$

Thus $\ker(Q_1 \cdot -)$ is the subspace generated by all the monomials $w_2^a w_3^b U_h$ where $a+b$ is odd, and $\mathrm{im}(Q_1 \cdot -)$ is the subspace generated by all the monomials $w_2^a w_3^{b+1} U_h$ where $a+b$

is even. Rephrased, $\mathrm{im}(Q_1 \cdot -)$ is the subspace generated by all the monomials $w_2^a w_3^b U_h$ where $a+b$ is odd and $b \geq 1$. It follows that $H_*(\mathbb{Z}/2\mathbb{Z}[w_2, w_3]U_h; Q_1) \cong \mathbb{Z}/2\mathbb{Z}[w_2^2]w_2 U_h$.

Equation 6.7 implies that the image of $Q_1 \cdot -$ on $\mathbb{Z}/2\mathbb{Z}[g_{2^r-2}]$ is trivial, so

$$H_*(\mathbb{Z}/2\mathbb{Z}[g_{2^r-2}]; Q_1) \cong \mathbb{Z}/2\mathbb{Z}[g_{2^r-2}].$$

Similarly, Equation 6.7 implies that $t_{2i+3} \in \mathrm{im}(Q_1 \cdot -)$ on $\mathbb{Z}/2\mathbb{Z}[w_{2i}, t_{2i+3}]$, while the kernel of $Q_1 \cdot -$ is generated by w_{2i}^2 and t_{2i+3}. It follows that

$$H_*(\mathbb{Z}/2\mathbb{Z}[w_{2i}, t_{2i+3}]; Q_1) \cong \mathbb{Z}/2\mathbb{Z}[w_{2i}^2],$$

and we are done. $\square$

6.3. **$\bar{\theta}$ induces isomorphisms on Q_i-homology.** Recall the map $\bar{\theta} : \overline{N} \to M_h$ from Notation 6.2. Our next goal is to prove that $\bar{\theta}$ induces isomorphisms $H_*(\overline{N}; Q_i) \to H_*(M_h; Q_i)$ for $i = 0$ and 1. We will do so by comparing $\bar{\theta}$ to the analogous map coming from the Anderson–Brown–Peterson splitting of MSpin^c.[7] For this, we need the following lemma relating $H_*(\text{¿}_{\mathcal{A}}; Q_i)$ and $H_*(E_{\mathcal{A}}; Q_i)$ to $H_*(C_{\mathcal{A}}; Q_i)$, where C is the $\mathcal{A}_1$-module defined in Definition 4.14.

Notation 6.18. Let $e_0 := \mathrm{Sq}^1$ and $e_1 := \mathrm{Sq}^2$ be the generators of $E_{\mathcal{A}}$. Let $q_0 := \mathrm{Sq}^1$ and $c_0 := \mathrm{Sq}^1$ denote the generators of $\text{¿}_{\mathcal{A}}$ and $C_{\mathcal{A}}$, respectively.

Lemma 6.19. *There are unique non-trivial maps $\text{¿}_{\mathcal{A}} \to C_{\mathcal{A}}$ and $E_{\mathcal{A}} \to C_{\mathcal{A}}$. Moreover, these maps induce monomorphisms on $H_*(-; Q_i)$.*

Proof. Note that the only non-zero element of $C_{\mathcal{A}}$ of degree zero is c_0, and that $C_{\mathcal{A}}$ has no non-zero elements of degree one. Thus if non-trivial maps $\text{¿}_{\mathcal{A}} \to C_{\mathcal{A}}$ and $E_{\mathcal{A}} \to C_{\mathcal{A}}$ exist, they must be given by $q_0 \mapsto c_0$ and

$$\begin{aligned} e_0 &\mapsto c_0, \\ e_1 &\mapsto 0, \end{aligned}$$

respectively. We will show that these determine maps of $\mathcal{A}_1$-modules, and tensoring with $\mathcal{A}$ will give the desired maps of $\mathcal{A}$-modules. Consider the map $\text{¿} \to C$ given by

$$\begin{aligned} q_0 &\mapsto c_0, \\ \mathrm{Sq}^2 q_0 &\mapsto \mathrm{Sq}^2 c_0, \\ \mathrm{Sq}^3 q_0 &\mapsto 0. \end{aligned}$$

This map commutes with the action of $\mathcal{A}_1$ on ¿ and C, so this is a map of $\mathcal{A}_1$-modules and therefore induces a map of $\mathcal{A}$-modules $\text{¿}_{\mathcal{A}} \to C_{\mathcal{A}}$.

[7] A posteriori, what makes this approach work is that we have a natural bijection of non-Eilenberg–Mac Lane summands in the 2-local splittings of MSpin^c and MSpin^h. For another consequence of this observation, see Corollary 8.6.

Next, consider the map $E \to C$ given by

$$\begin{aligned} e_0 &\mapsto c_0, \\ e_1 &\mapsto 0, \\ \mathrm{Sq}^1 e_1 &\mapsto 0, \\ \mathrm{Sq}^2 e_0 &\mapsto \mathrm{Sq}^2 c_0, \end{aligned}$$

and sending all elements of higher degree to zero. As before, this map commutes with the action of $\mathcal{A}_1$ on E and C, so this determines a map of $\mathcal{A}_1$-modules and hence induces the desired $\mathcal{A}$-module map $E_{\mathcal{A}} \to C_{\mathcal{A}}$.

To see that these maps induce injections in homology, notice that both maps $Q \to C$ and $E \to C$ are surjective. The only element in the kernel of $Q \to C$ is $\mathrm{Sq}^3 q_0$, so the kernel is isomorphic to $\Sigma^3 \mathcal{A}_1/(\mathcal{A}_1\mathrm{Sq}^1 + \mathcal{A}_1\mathrm{Sq}^2)$. This yields a short exact sequence

$$0 \longrightarrow \Sigma^3 \mathcal{A}_1/(\mathcal{A}_1\mathrm{Sq}^1 + \mathcal{A}_1\mathrm{Sq}^2) \longrightarrow \mathcal{Q} \longrightarrow C \longrightarrow 0. \tag{6.8}$$

Tensoring Equation 6.8 with $\mathcal{A}$ gives us a short exact sequence

$$0 \longrightarrow \Sigma^3 \mathcal{A}/(\mathcal{A}\mathrm{Sq}^1 + \mathcal{A}\mathrm{Sq}^2) \longrightarrow \mathcal{Q}_{\mathcal{A}} \longrightarrow C_{\mathcal{A}} \longrightarrow 0. \tag{6.9}$$

Equation 6.9 induces a long exact sequence in Q_i-homology. The relevant part is

$$H_j(\Sigma^3 \mathcal{A}/(\mathcal{A}\mathrm{Sq}^1 + \mathcal{A}\mathrm{Sq}^2); Q_i) \longrightarrow H_j(\mathcal{Q}_{\mathcal{A}}; Q_i) \longrightarrow H_j(C_{\mathcal{A}}; Q_i).$$

Showing that the map $H_j(\mathcal{Q}_{\mathcal{A}}; Q_i) \to H_j(C_{\mathcal{A}}; Q_i)$ is injective is equivalent to showing that

$$H_j(\Sigma^3 \mathcal{A}/(\mathcal{A}\mathrm{Sq}^1 + \mathcal{A}\mathrm{Sq}^2); Q_i) \to H_j(\mathcal{Q}_{\mathcal{A}}; Q_i) \tag{6.10}$$

is zero. Since there is an isomorphism $H_j(\Sigma^3 \mathcal{A}/(\mathcal{A}\mathrm{Sq}^1 + \mathcal{A}\mathrm{Sq}^2); Q_i) \cong H_{j-3}(\mathcal{A}/(\mathcal{A}\mathrm{Sq}^1 + \mathcal{A}\mathrm{Sq}^2); Q_i)$ and the homology $H_*(\mathcal{A}/(\mathcal{A}\mathrm{Sq}^1 + \mathcal{A}\mathrm{Sq}^2); Q_i)$ is only nonzero in even degrees (Lemma 6.5), the map in Equation 6.10 can only be nonzero for j odd. But $H_j(\mathcal{Q}_{\mathcal{A}}; Q_i) \cong 0$ for j odd (Proposition 6.8), so $H_j(\mathcal{Q}_{\mathcal{A}}; Q_i) \to H_j(C_{\mathcal{A}}; Q_i)$ is injective.

The argument for E is similar. The kernel of $E \to C$ is isomorphic to $\Sigma \mathcal{A}_1/\mathcal{A}_1\mathrm{Sq}^3$, with the inclusion $\Sigma \mathcal{A}_1/\mathcal{A}_1\mathrm{Sq}^3 \to E$ given by $1 \mapsto e_1$. Tensoring by $\mathcal{A}$ gives us a short exact sequence

$$0 \longrightarrow \Sigma \mathcal{A}/\mathcal{A}\mathrm{Sq}^3 \longrightarrow E_{\mathcal{A}} \longrightarrow C_{\mathcal{A}} \longrightarrow 0,$$

which induces an exact sequence

$$H_j(\Sigma \mathcal{A}/\mathcal{A}\mathrm{Sq}^3; Q_i) \longrightarrow H_j(E_{\mathcal{A}}; Q_i) \longrightarrow H_j(C_{\mathcal{A}}; Q_i).$$

Again, it will suffice to show that the the map $H_j(\Sigma \mathcal{A}/\mathcal{A}\mathrm{Sq}^3; Q_i) \to H_j(E_{\mathcal{A}}; Q_i)$ is zero. There is an isomorphism $H_j(\Sigma \mathcal{A}/\mathcal{A}\mathrm{Sq}^3; Q_i) \cong H_{j-1}(\mathcal{A}/\mathcal{A}\mathrm{Sq}^3; Q_i)$ and $H_*(\mathcal{A}/\mathcal{A}\mathrm{Sq}^3; Q_i)$ is nonzero only in even degrees (Lemma 6.5), so this map has a non-zero domain only when j is odd. But $H_*(E_{\mathcal{A}}; Q_i)$ is zero in odd degrees (Proposition 6.9), so the codomain is trivial if j is odd. Hence the map is always zero. $\square$

Setup 6.20. We now explain the comparison to MSpin^c that we will use to compute $\bar{\theta}$ on Q_i-homology. Let $\overline{N}_c = \bigoplus_{I \in \mathcal{P}} \Sigma^{4|I|} C_{\mathcal{A}}$. Then we have the commutative diagram

$$\begin{tikzcd}
\overline{N}_c \arrow[d] \arrow[drr, "\bar{\theta}_c"] & & \\
\bigoplus_{I \in \mathcal{P}} \Sigma^{4|I|} C_{\mathcal{A}} \oplus \bigoplus_{z \in Z_c} \Sigma^{\deg z} \mathcal{A} \arrow[rr, "\psi"', "\cong"] & & M_c,
\end{tikzcd}$$

where the vertical map is the inclusion of the summands on the left and ψ is the isomorphism in cohomology induced by the Anderson–Brown–Peterson splitting of MSpin^c. Since ψ is an isomorphism, it induces isomorphisms on Q_i-homology. The vertical map induces isomorphisms on Q_i-homology because the Q_i-homology of each $\Sigma^{\deg z} \mathcal{A}$ summand vanishes [Mar83, p. 331, Proposition 1]. It follows that $\bar{\theta}_c$ induces isomorphisms on $H_*(-; Q_i)$. Moreover, $\bar{\theta}_c$ takes the generator c_0 of the $C_{\mathcal{A}}$ summand corresponding to a partition $I \in \mathcal{P}$ to $p_I U_c \in M_c$, where $U_c \in M_c := H^*\mathrm{MSpin}^c$ is the Thom class.

Altogether, we have a diagram

$$\begin{tikzcd}
\overline{N} \arrow[d, "\bar{\theta}"] & \overline{N}_c \arrow[d, "\bar{\theta}_c"] \\
M_h \arrow[r] & M_c
\end{tikzcd} \tag{6.11}$$

in the category of $\mathcal{A}$-modules. If we could fill this in to make a commuting square, then understanding the map $M_h \to M_c$ would give us control over $\bar{\theta}$. Unfortunately, there is no obvious way to do this. Instead, we will fill in Diagram 6.11 to a non-commutative diagram that yields a commutative diagram on Q_i-homology.

We define a map $\overline{N} \to \overline{N}_c$ by treating even partition summands and odd partition summands separately. For $I \in \mathcal{P}_{\mathrm{even}}$, set

$$\begin{aligned} \Sigma^{4|I|} J_{\mathcal{A}} &\to \Sigma^{4|I|} C_{\mathcal{A}} \\ q_0 &\mapsto c_0. \end{aligned}$$

For $I \in \mathcal{P}_{\mathrm{odd}}$, set

$$\begin{aligned} \Sigma^{4|I|} E_{\mathcal{A}} &\to \Sigma^{4|I|} C_{\mathcal{A}} \\ e_0 &\mapsto c_0, \\ e_1 &\mapsto 0. \end{aligned}$$

Combined with Diagram 6.11, this gives us a non-commuting square

$$\begin{tikzcd}
\overline{N} \arrow[r] \arrow[d, "\bar{\theta}"] & \overline{N}_c \arrow[d, "\bar{\theta}_c"] \\
M_h \arrow[r] & M_c.
\end{tikzcd} \tag{6.12}$$

Remark 6.21. Note that Diagram 6.12 does commute when we restrict $\overline{N}$ to the submodule generated by all the $E_{\mathcal{A}}$ summands. Indeed, the top arrow followed by $\bar{\theta}_c$ takes

$e_0 \mapsto p_I U_c$ and $e_1 \mapsto 0$, and $\bar{\theta}$ followed by the bottom arrow takes $e_0 \mapsto p_I U_c$ and e_1 to the product of the image of β_I and $w_3 U_c$. Since w_3 vanishes in $H^*\mathrm{BSpin}^c$, we find that $e_1 \mapsto 0$.

Next up, we show that $\bar{\theta}$ induces an injection on Q_0-homology.

Lemma 6.22. *The map $\bar{\theta} : \overline{N} \to M_h$ induces an injection on Q_0-homology.*

Proof. Applying $H_*(-;Q_0)$ to Diagram 6.12 gives us the diagram

$$\begin{tikzcd} H_*(\overline{N};Q_0) \arrow[r] \arrow[d, "\bar{\theta}_*"] & H_*(\overline{N}_c;Q_0) \arrow[d, "\bar{\theta}_{c*}"] \\ H_*(M_h;Q_0) \arrow[r] & H_*(M_c;Q_0) \end{tikzcd} \tag{6.13}$$

of $\mathbb{Z}/2\mathbb{Z}$-vector spaces. We claim that Diagram 6.13 commutes, which we will prove by checking commutativity for each generator of $H_*(\overline{N};Q_0)$. We will then use our understanding of $\bar{\theta}_{c*}$ to prove that $\bar{\theta}_*$ is an isomorphism.

If I is an even partition, then the generator $\chi(\mathrm{Sq}^{4k})q_0 \in H_*(¿_{\mathcal{A}};Q_0)$ satisfies

$$\begin{array}{ccccc} H_*(\overline{N};Q_0) & \longrightarrow & H_*(\overline{N}_c;Q_0) & \xrightarrow{\bar{\theta}_{c*}} & H_*(M_c;Q_0) \\ \chi(\mathrm{Sq}^{4k})q_0 & \longmapsto & \chi(\mathrm{Sq}^{4k})c_0 & \longmapsto & \chi(\mathrm{Sq}^{4k})p_I U_c. \end{array}$$

Taking the other path around Diagram 6.13, Proposition 5.21 implies

$$\begin{array}{ccccc} H_*(\overline{N};Q_0) & \xrightarrow{\quad\bar{\theta}_*\quad} & H_*(M_h;Q_0) & \longrightarrow & H_*(M_c;Q_0) \\ \chi(\mathrm{Sq}^{4k})q_0 & \longmapsto & \chi(\mathrm{Sq}^{4k})(p_I + \mathrm{Sq}^3\mathrm{Sq}^1\alpha_I)U_h. & & \end{array}$$

We need to show that $\chi(\mathrm{Sq}^{4k})(p_I+\mathrm{Sq}^3\mathrm{Sq}^1\alpha_I)U_h$ maps to $\chi(\mathrm{Sq}^{4k})p_I U_c$. We first calculate

$$\begin{aligned} \Delta(\mathrm{Sq}^3\mathrm{Sq}^1) &= (\Delta\mathrm{Sq}^3)(\Delta\mathrm{Sq}^1) \\ &= (\mathrm{Sq}^3 \otimes 1 + \mathrm{Sq}^2 \otimes \mathrm{Sq}^1 + \mathrm{Sq}^1 \otimes \mathrm{Sq}^2 + 1 \otimes \mathrm{Sq}^3)(\mathrm{Sq}^1 \otimes 1 + 1 \otimes \mathrm{Sq}^1) \\ &= \mathrm{Sq}^3\mathrm{Sq}^1 \otimes 1 + \mathrm{Sq}^2\mathrm{Sq}^1 \otimes \mathrm{Sq}^1 + \mathrm{Sq}^1\mathrm{Sq}^1 \otimes \mathrm{Sq}^2 + \mathrm{Sq}^1 \otimes \mathrm{Sq}^3 \\ &\quad + \mathrm{Sq}^3 \otimes \mathrm{Sq}^1 + \mathrm{Sq}^2 \otimes \mathrm{Sq}^1\mathrm{Sq}^1 + \mathrm{Sq}^1 \otimes \mathrm{Sq}^2\mathrm{Sq}^1 + 1 \otimes \mathrm{Sq}^3\mathrm{Sq}^1 \\ &= \mathrm{Sq}^3\mathrm{Sq}^1 \otimes 1 + 1 \otimes \mathrm{Sq}^3\mathrm{Sq}^1 + \mathrm{Sq}^3 \otimes \mathrm{Sq}^1 + \mathrm{Sq}^1 \otimes \mathrm{Sq}^3 \\ &\quad + \mathrm{Sq}^2\mathrm{Sq}^1 \otimes \mathrm{Sq}^1 + \mathrm{Sq}^1 \otimes \mathrm{Sq}^2\mathrm{Sq}^1. \end{aligned}$$

From this, we calculate

$$\begin{aligned} \mathrm{Sq}^3\mathrm{Sq}^1(\alpha_I U_h) &= (\mathrm{Sq}^3\mathrm{Sq}^1\alpha_I)U_h + (\mathrm{Sq}^1\alpha_I)w_3 U_h \\ &= (\mathrm{Sq}^3\mathrm{Sq}^1\alpha_I)U_h + \mathrm{Sq}^1(\alpha_I w_3 U_h), \end{aligned}$$

so

$$(\mathrm{Sq}^3\mathrm{Sq}^1\alpha_I)U_h = \mathrm{Sq}^3\mathrm{Sq}^1(\alpha_I U_h) + \mathrm{Sq}^1(\alpha_I w_3 U_h). \tag{6.14}$$

It follows that $\chi(\mathrm{Sq}^{4k})q_0$ maps to $\chi(\mathrm{Sq}^{4k})(p_I U_h + \mathrm{Sq}^3\mathrm{Sq}^1(\alpha_I U_h) + \mathrm{Sq}^1(\alpha_I w_3 U_h))$.

To deal with the $\chi(\mathrm{Sq}^{4k}\mathrm{Sq}^3\mathrm{Sq}^1(\alpha_I U_h)$ term, we check that $\chi(\mathrm{Sq}^{4k}\mathrm{Sq}^3\mathrm{Sq}^1)$ is a Q_0-cycle in $\mathcal{A}$:

$$\begin{aligned}
\mathrm{Sq}^1\chi(\mathrm{Sq}^{4k})\mathrm{Sq}^3\mathrm{Sq}^1 &= \chi(\mathrm{Sq}^1)\chi(\mathrm{Sq}^{4k})\mathrm{Sq}^3\mathrm{Sq}^1 \\
&= \chi(\mathrm{Sq}^{4k}\mathrm{Sq}^1)\mathrm{Sq}^3\mathrm{Sq}^1 \\
&= \chi(\mathrm{Sq}^1\mathrm{Sq}^{4k} + Q_1\mathrm{Sq}^{4k-2})\mathrm{Sq}^3\mathrm{Sq}^1 \\
&= \chi(\mathrm{Sq}^{4k})\mathrm{Sq}^1\mathrm{Sq}^3\mathrm{Sq}^1 + \chi(\mathrm{Sq}^{4k-2})Q_1\mathrm{Sq}^3\mathrm{Sq}^1 \\
&= 0.
\end{aligned}$$

Since the Q_0-homology of $\mathcal{A}$ vanishes, there is some $a \in \mathcal{A}$ such that $\chi(\mathrm{Sq}^{4k})\mathrm{Sq}^3\mathrm{Sq}^1 = \mathrm{Sq}^1 a$. Thus $\chi(\mathrm{Sq}^{4k})\mathrm{Sq}^3\mathrm{Sq}^1(\alpha_I U_h) = \mathrm{Sq}^1 a(\alpha_I U_h)$, and this term vanishes in Q_0-homology.

At this point, we have deduced that $\chi(\mathrm{Sq}^{4k})q_0$ maps to the element $\chi(\mathrm{Sq}^{4k})(p_I U_h + \mathrm{Sq}^1(\alpha_I w_3 U_h))$. To deal with the $\chi(\mathrm{Sq}^{4k})\mathrm{Sq}^1(\alpha_I w_3 U_h)$ term, recall that w_3 vanishes in $H^*\mathrm{BSpin}^c$. Thus $\chi(\mathrm{Sq}^{4k})\mathrm{Sq}^1(\alpha_I w_3 U_h)$ maps to zero in M_c, and we find that $\chi(\mathrm{Sq}^{4k})q_0$ maps to $\chi(\mathrm{Sq}^{4k})(p_I U_c)$, as desired. Thus Diagram 6.13 commutes for each $\text{¿}_{\mathcal{A}}$ summand. Since Diagram 6.12 commutes for the $E_{\mathcal{A}}$ summands, Diagram 6.13 commutes for the $E_{\mathcal{A}}$ summands. Thus Diagram 6.13 commutes.

Because $\bar{\theta}_{c*}$ is an isomorphism by Theorem 3.6, and since the map

$$H_*(\overline{N}; Q_0) \to H_*(\overline{N}_c; Q_0)$$

is the direct sum of injective maps (and is hence injective), the composite $H_*(\overline{N}; Q_0) \to H_*(M_c; Q_0)$ is injective. This implies that $\bar{\theta}_*$ is injective too. ☐

Now we strengthen Lemma 6.22 by showing that $\bar{\theta}_*$ is in fact an isomorphism.

Lemma 6.23. *The map $\bar{\theta} : \overline{N} \to M_h$ induces an isomorphism on Q_0-homology.*

Proof. Since $\bar{\theta}_*$ is injective, we just need to show that dimensions of $H_*(\overline{N}; Q_0)$ and $H_*(M_h; Q_0)$ in each degree are equal. We will prove that these dimensions are equal in each degree by showing that $H_*(\overline{N}; Q_0)$ and $H_*(M_h; Q_0)$ have the same Hilbert–Poincaré series.

First, we compute the Hilbert–Poincaré series of $H_*(M_h; Q_0)$. By Lemma 6.15, we can write any monomial in $H_*(M_h; Q_0)$ as ABU_h, where A is a monomial in the w_{2i}^2 and $f_{2^k}^2$ and B is a product of f_{2^k} with each factor occurring at most once. The number of monomials of the form A in degree $4n$ is the number of partitions of n, and there are no monomials of this form in degrees not divisible by four. Since each f_{2^k} has degree 2^k (ranging over $k \geq 2$), the degree of B is the number whose binary expansion has a 1 in the k^{th} place for every f_{2^k} factor. There is one such B for every number divisible by four, and no others. Hence the Hilbert–Poincaré series of $H_*(M_h; Q_0)$ is $\sum_{I \in \mathcal{P}} t^{4|I|}(1 - t^4)^{-1}$.

The Hilbert–Poincaré series for $H_*(\overline{N}; Q_0)$ is simple to compute: the Hilbert–Poincaré series of both $H_*(\text{¿}_{\mathcal{A}}; Q_0)$ and $H_*(E_{\mathcal{A}}; Q_0)$ is $(1 - t^4)^{-1}$ (Propositions 6.8 and 6.9), and for each partition I, there is a single summand of $\text{¿}_{\mathcal{A}}$ or $E_{\mathcal{A}}$ shifted by degree $4|I|$. Thus the Hilbert–Poincaré series of $H_*(\overline{N};Q_0)$ is also $\sum_{I\in\mathcal{P}} t^{4|I|}(1-t^4)^{-1}$. □

Now we turn to the effect of $\bar{\theta}$ on Q_1-homology, employing the same strategies as before. We will again see that $\bar{\theta}_*$ is injective and even an isomorphism.

Lemma 6.24. *The map $\bar{\theta}:\overline{N}\to M_h$ induces an injection on Q_1-homology.*

Proof. As in Lemma 6.22, we will show that Diagram 6.12 commutes after applying the functor $H_*(-;Q_1)$ by checking on generators. If I is an even partition, then the generator $\chi(\mathrm{Sq}^{2\sum_{\ell=1}^k \Delta_{i_\ell}})\mathrm{Sq}^2 q_0 \in H_*(\text{¿}_{\mathcal{A}};Q_1)$ satisfies

$$\begin{array}{ccccc} H_*(\overline{N};Q_1) & \longrightarrow & H_*(\overline{N}_c;Q_1) & \xrightarrow{\bar{\theta}_{c*}} & H_*(M_c;Q_1) \\ \chi(\mathrm{Sq}^{2\sum_{\ell=1}^k \Delta_{i_\ell}})\mathrm{Sq}^2 q_0 & \longmapsto & \chi(\mathrm{Sq}^{2\sum_{\ell=1}^k \Delta_{i_\ell}})\mathrm{Sq}^2 c_0 & \longmapsto & \chi(\mathrm{Sq}^{2\sum_{\ell=1}^k \Delta_{i_\ell}})\mathrm{Sq}^2(p_I U_c). \end{array}$$

For the other path (in Diagram 6.15), we get

$$\bar{\theta}_*(\chi(\mathrm{Sq}^{2\sum_{\ell=1}^k \Delta_{i_\ell}})\mathrm{Sq}^2 q_0) = \chi(\mathrm{Sq}^{2\sum_{\ell=1}^k \Delta_{i_\ell}})\mathrm{Sq}^2(p_I U_h + (\mathrm{Sq}^3\mathrm{Sq}^1\alpha_I)U_h).$$

By Equation 6.14, this maps to

$$\chi(\mathrm{Sq}^{2\sum_{\ell=1}^k \Delta_{i_\ell}})(\mathrm{Sq}^2(p_I U_h) + \mathrm{Sq}^2\mathrm{Sq}^3\mathrm{Sq}^1(\alpha_I U_h) + \mathrm{Sq}^2\mathrm{Sq}^1(\alpha_I w_3 U_h)).$$

We then compute

$$\begin{aligned} Q_1\chi(\mathrm{Sq}^{2\sum_{\ell=1}^k \Delta_{i_\ell}})\mathrm{Sq}^2\mathrm{Sq}^3\mathrm{Sq}^1 &= \chi(Q_1)\chi(\mathrm{Sq}^{2\sum_{\ell=1}^k \Delta_{i_\ell}})\mathrm{Sq}^2\mathrm{Sq}^3\mathrm{Sq}^1 \\ &= \chi(\mathrm{Sq}^{2\sum_{\ell=1}^k \Delta_{i_\ell}} Q_1)\mathrm{Sq}^2\mathrm{Sq}^3\mathrm{Sq}^1 \\ &= \chi(Q_1\mathrm{Sq}^{2\sum_{\ell=1}^k \Delta_{i_\ell}})\mathrm{Sq}^2\mathrm{Sq}^3\mathrm{Sq}^1 \\ &= \chi(\mathrm{Sq}^{2\sum_{\ell=1}^k \Delta_{i_\ell}})Q_1\mathrm{Sq}^2\mathrm{Sq}^3\mathrm{Sq}^1 \\ &= 0. \end{aligned}$$

Since $H_*(\mathcal{A};Q_1)\cong 0$, this implies there is some $a\in A$ with $\chi(\mathrm{Sq}^{2\sum_{\ell=1}^k \Delta_{i_\ell}})\mathrm{Sq}^3\mathrm{Sq}^1 = Q_1 a$. Hence the term $\chi(\mathrm{Sq}^{2\sum_{\ell=1}^k \Delta_{i_\ell}})\mathrm{Sq}^2\mathrm{Sq}^3\mathrm{Sq}^1(\alpha_I U_h)$ is a boundary and vanishes in Q_1-homology, and therefore our generator maps to

$$\chi(\mathrm{Sq}^{2\sum_{\ell=1}^k \Delta_{i_\ell}})(\mathrm{Sq}^2(p_I U_h) + \mathrm{Sq}^2\mathrm{Sq}^1(\alpha_I w_3 U_h)) \in H_*(M_h;Q_1).$$

As w_3 vanishes in $H^*\mathrm{BSpin}^c$, this maps to

$$\chi(\mathrm{Sq}^{2\sum_{\ell=1}^k \Delta_{i_\ell}})\mathrm{Sq}^2(p_I U_c) \in H_*(M_c;Q_1).$$

Hence the diagram

$$\begin{tikzcd} H_*(\overline{N};Q_1) \arrow[r] \arrow[d, "\bar{\theta}_*"] & H_*(\overline{N}_c;Q_1) \arrow[d, "\bar{\theta}_{c*}"] \\ H_*(M_h;Q_1) \arrow[r] & H_*(M_c;Q_1) \end{tikzcd} \tag{6.15}$$

commutes for the $\mathcal{E}_{\mathcal{A}}$ summands. Diagram 6.12 commutes on the $E_{\mathcal{A}}$ summands, so Diagram 6.15 commutes for the $E_{\mathcal{A}}$ summands. Thus Diagram 6.15 commutes in general. Theorem 3.6 implies that $\bar{\theta}_{c*}$ is an isomorphism, and $H_*(\overline{N}; Q_1) \to H_*(\overline{N}_c; Q_1)$ is a direct sum of injective maps (and is hence injective), so $\bar{\theta}_*$ must be injective as well. □

Lemma 6.25. *The map $\bar{\theta} : \overline{N} \to M_h$ induces an isomorphism on Q_1-homology.*

Proof. As in Lemma 6.23, it suffices to show that $H_*(M_h; Q_1)$ and $H_*(\overline{N}; Q_1)$ have the same Hilbert–Poincaré series. By Lemma 6.17, we can write any monomial in $H_*(M_h; Q_1)$ as ABw_2U_h, where A is a monomial in the w_{2i}^2 and $g_{2^r-2}^2$ and B is a product of g_{2^r-2} with each factor occurring at most once. The number of monomials of the form A in degree $4n$ is the number of partitions of n, and there are no monomials of this form in degrees not divisible by four. Let s be the Hilbert–Poincaré series for the exterior algebra $\bigwedge[g_{2^r-2} \mid r \geq 3]$. Then the Hilbert–Poincaré series of $H_*(M_h; Q_1)$ is $t^2 \sum_{I \in \mathcal{P}} t^{4|I|} s$.

For the Hilbert–Poincaré series of $H_*(\overline{N}; Q_1)$, note that the degree of $\chi(\mathrm{Sq}^{2\sum_{\ell=1}^{k} \Delta_{i_\ell}})$ for $i_1 > \ldots > i_k \geq 2$ is

$$2\sum_{\ell=1}^{k}(2^{i_\ell} - 1) = \sum_{\ell=1}^{k}(2^{i_\ell+1} - 2),$$

which is also the degree of the exterior product $g_{2^{i_1+1}-2} \cdots g_{2^{i_k+1}-2}$. So the Hilbert–Poincaré series of $H_*(\mathcal{E}_{\mathcal{A}}; Q_1)$ and $H_*(E_{\mathcal{A}}; Q_1)$ are both $t^2 s$, and hence the Hilbert–Poincaré series of $H_*(\overline{N}; Q_1)$ is $\sum_{I \in \mathcal{P}} t^{4|I|} t^2 s$, as desired. □

Corollary 6.26. *The map $\bar{\theta} : \overline{N} \to M_h$ induces isomorphisms on Q_i-homology.*

Proof. This is just the combination of Lemma 6.23 and Lemma 6.25. □

7. Anderson–Brown–Peterson splitting of MSpin^h

Using the Q_i-homology isomorphisms given in Section 6, we now prove Theorem 1.1 (which we restate here for convenience).

Theorem 7.1. *There is a set of homogeneous classes $Z \subset H^*\mathrm{MSpin}^h$ and a map*

$$\mathrm{MSpin}^h \to \bigvee_{I \in \mathcal{P}_{\mathrm{even}}} \mathrm{ksp}\langle 4|I| \rangle \vee \bigvee_{I \in \mathcal{P}_{\mathrm{odd}}} \Sigma^{4|I|} F \vee \bigvee_{z \in Z} \Sigma^{\deg z} H\mathbb{Z}/2\mathbb{Z}$$

that is a 2-local homotopy equivalence.

To begin, we need to construct our class $Z \subset H^*\mathrm{MSpin}^h$ of homogeneous classes. As before, we will use the notation $M_h := H^*\mathrm{MSpin}^h$.

Setup 7.2. Let $\mathcal{A}_+ \subset \mathcal{A}$ be the (left) submodule generated by all elements of positive degree. Now form the composition

$$\overline{N} \xrightarrow{\bar{\theta}} M_h \xrightarrow{\rho} M_h/\mathcal{A}_+ M_h,$$

where $\bar{\theta}$ is the map given in Notation 6.2 and $\rho : M_h \to M_h/\mathcal{A}_+M_h$ is the quotient map. Take the cokernel $c : M_h/\mathcal{A}_+M_h \to R$ of $\rho \circ \bar{\theta}$. Let $Z \subset M_h$ be any collection of homogeneous elements such $c \circ \rho(Z)$ is a basis for R.

We will show that Z is (an instance of) the desired set of homogeneous classes. In order to prove Theorem 7.1, we first need to expand $\overline{N}$ to include the cohomology of $\bigvee_{z\in Z} \Sigma^{\deg z} H\mathbb{Z}/2\mathbb{Z}$.

Notation 7.3. Define
$$N := \overline{N} \oplus \bigoplus_{z\in Z} \Sigma^{\deg z}\mathcal{A},$$
and let $\theta : N \to M_h$ be the map defined by $\bar{\theta}$ on $\overline{N}$ and the maps
$$\begin{aligned} \Sigma^{\deg z}\mathcal{A} &\to M_h \\ 1 &\mapsto z \end{aligned}$$
for each $z \in Z$.

Surjectivity of θ is relatively straightforward.

Lemma 7.4. *The map $\theta : N \to M_h$ is surjective.*

Proof. Suppose $x \in M_h$. Then there are $z_1, \ldots z_n \in Z$ such that $c\rho x = c\rho z_1 + \ldots + c\rho z_n$, since $c\rho(Z)$ forms a basis of R. Thus there exists $y \in \overline{N}$ such that $\rho x = \rho z_1 + \ldots + \rho z_n + \rho\bar{\theta}y$, and there is some $a \in \mathcal{A}_+$ and $x' \in M_h$ such that $x = z_1 + \ldots + z_n + \bar{\theta}y + ax'$. In particular, x' has lower degree than x. Since M_h is bounded below (in degree), we can repeat this procedure until x is written as a sum of Steenrod squares of elements of Z and elements of $\bar{\theta}(\overline{N})$. Hence x is in the image of θ. □

Showing that θ is injective requires more work. The idea is to filter N and M_h and show that θ induces an isomorphism at each step in the filtration.

Notation 7.5. For $n \in \mathbb{Z}$, let
$$N^{[n]} \subset N$$
be the submodule given by the direct sum of all the $¿_{\mathcal{A}}$, $E_{\mathcal{A}}$, and $\mathcal{A}$ summands that are non-zero in degrees less than or equal to n. Let
$$M_h^{[n]} := \theta(N^{[n]}).$$
Denote the restriction of θ by $\theta_n : N^{[n]} \to M_h^{[n]}$, and let $\lambda_n : N/N^{[n-1]} \to M_h/M_h^{[n-1]}$ be the induced map on quotients.

Note that by our definition of $N^{[n-1]}$, the module $N/N^{[n-1]}$ is the direct sum of those summands of N that are zero in degrees less than n, and $N^{[n]}/N^{[n-1]}$ is the direct sum of those summands that are zero in degrees less than n but nonzero in degree n. Also, each summand of N is of the form $B_{\mathcal{A}}$ for some $\mathcal{A}_1$-module B (e.g. the free summands of N take the form $\mathcal{A} \cong (\mathcal{A}_1)_{\mathcal{A}}$).

Definition 7.6. Define $P_n \subset N^{[n]}/N^{[n-1]}$ to be the $\mathcal{A}_1$-submodule given by the direct sum of B for each summand $B_{\mathcal{A}}$ of $N^{[n]}/N^{[n-1]}$.

Lemma 7.7. *If $\theta_{n-1} : N^{[n-1]} \to M_h^{[n-1]}$ is an isomorphism, then the restriction of λ_n to P_n is injective.*

Proof. First, notice that P_n can be written as $X_n \oplus Y_n \oplus Z_n$, where $X_n = \bigoplus_{\alpha \in A_X} \Sigma^n \text{¿}$, $Y_n = \bigoplus_{\alpha \in A_Y} \Sigma^n E$, and $Z_n = \bigoplus_{\alpha \in A_Z} \Sigma^n \mathcal{A}_1$, and A_X, A_Y, and A_Z are finite sets. Moreover, A_X is non-empty only if $n = 0 \pmod 8$ and A_Y is non-empty only if $n = 4 \pmod 8$. In particular, one or the other is empty. We have short exact sequences fitting in commutative diagrams

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & N^{[n-1]} \arrow[r] \arrow[d, "\theta_{n-1}"] & N \arrow[r] \arrow[d, "\theta"] & N/N^{[n-1]} \arrow[r] \arrow[d, "\lambda_n"] & 0 \arrow[d] \\
0 \arrow[r] & M_h^{[n-1]} \arrow[r] & M_h \arrow[r] & M_h/M_h^{[n-1]} \arrow[r] & 0,
\end{tikzcd}$$

so we get long exact sequences in Q_i-homology:

$$\begin{tikzcd}
H_j(N^{[n-1]}; Q_i) \arrow[r] \arrow[d, "\theta_{n-1*}"] & H_j(N; Q_i) \arrow[r] \arrow[d, "\theta_*"] & H_j(N/N^{[n-1]}; Q_i) \arrow[r] \arrow[d, "\lambda_{n*}"] & {} \\
H_j(M_h^{[n-1]}; Q_i) \arrow[r] & H_j(M_h; Q_i) \arrow[r] & H_j(M_h/M_h^{[n-1]}; Q_i) \arrow[r] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & H_{j+\deg Q_i}(N^{[n-1]}; Q_i) \arrow[r] \arrow[d, "\theta_{n-1*}"] & H_{j+\deg Q_i}(N; Q_i) \arrow[d, "\theta_*"] \\
{} \arrow[r] & H_{j+\deg Q_i}(M_h^{[n-1]}; Q_i) \arrow[r] & H_{j+\deg Q_i}(M_h; Q_i).
\end{tikzcd}$$

Each θ_{n-1*} is an isomorphism because θ_{n-1} is an isomorphism by hypothesis. Each θ_* is an isomorphism because $\bar{\theta}_*$ is an isomorphism by Corollary 6.26, and the inclusion $\overline{N} \to N$ induces isomorphisms on Q_i-homology because the Q_i-homology of each free (i.e. $\mathcal{A}$) summand vanishes. So by the five lemma, $\lambda_{n*} : H_i(N/N^{[n-1]}; Q_i) \to H_i(M_h/M_h^{[n-1]}; Q_i)$ is an isomorphism.

Now, to show that the restriction of λ_n to P_n is injective, note that the modules ¿, E, and $\mathcal{A}_1$ are concentrated in degrees 0 through 6, so P_n is concentrated in degrees n through $n+6$. It thus suffices to show that if $v \in P_n$ is homogeneous of degree $n+s$ for $0 \le s \le 6$, and if $\lambda_n v = 0$, then $v = 0$. We will describe the proof for $s = 0$. The proofs for $1 \le s \le 6$ are similar.

Suppose v has degree n. Then we can write $v = x + y + z$ for $x \in X_n$, $y \in Y_n$, and $z \in Z_n$. Setup 7.2 gives us a diagram

$$N \xrightarrow{\theta} M_h \xrightarrow{\rho} M_h/\mathcal{A}_+ M_h \xrightarrow{c} R \tag{7.1}$$

in which the element $v \in N$ maps to zero in M_h. But θz is a linear combination of elements of Z that map to basis vectors for R in Equation 7.1. Moreover, $c(x) = c(y) = 0$, so $z = 0$ (otherwise θz would give a relation among the basis vectors of R).

So $v = x+y$ and $x = 0$ or $y = 0$. In either case, if $v \neq 0$, then v represents a nonzero class in Q_0-homology, so the assumption $\lambda_n v = 0$ contradicts our previous conclusion that λ_n induces an isomorphism on Q_i-homology. Thus $v = 0$, and hence λ_n is a monomorphism on the degree n part of P_n. ☐

Using the structure of MSpin^h as a module over MSpin, we can strengthen Lemma 7.7 by extending the submodule on which λ_n is injective.

Lemma 7.8. *If θ_{n-1} is an isomorphism, then the restriction of λ_n to $N^{[n]}/N^{[n-1]}$ is injective.*

Proof. Since MSpin^h is a module spectrum over MSpin, taking cohomology gives $M_h := H^*\mathrm{MSpin}^h$ the structure of a comodule over the coalgebra $M := H^*\mathrm{MSpin}$. Specifically, the comultiplication $\mu : M_h \to M \otimes M_h$ is induced by the multiplication map $\mathrm{MSpin} \wedge \mathrm{MSpin}^h \to \mathrm{MSpin}^h$. The identity axiom for a comodule states that the diagram

$$\begin{tikzcd} M_h \arrow[r, "\mu"] \arrow[d, "\mathrm{id}_{M_h}"] & M \otimes M_h \arrow[d, "\epsilon\otimes\mathrm{id}_{M_h}"] \\ M_h & \mathbb{Z}/2\mathbb{Z} \otimes M_h \arrow[l] \end{tikzcd}$$

commutes, where $\mathbb{Z}/2\mathbb{Z} \otimes M_h \to M_h$ is the canonical isomorphism and $\epsilon : M \to \mathbb{Z}/2\mathbb{Z}$ is the map induced by the unit map $\mathbb{S} \to \mathrm{MSpin}$. Since the Thom class $U \in M$ is the only nonzero element of degree 0, we see that $\epsilon(U) = 1$ and $\epsilon(x) = 0$ if x has degree greater than zero. It follows that for any homogeneous $m \in M_h$, we have

$$\mu m = U \otimes m + \sum_{i=1}^{\alpha} \ell_i \otimes m_i. \tag{7.2}$$

Here m_i has degree strictly less than that of m, as ℓ_i has degree strictly greater than zero. Indeed, if $\mu m = \sum_{i=1}^{\alpha'} \ell'_i \otimes m'_i$, then the diagram above implies

$$m = \sum_{i=1}^{\alpha} \epsilon(\ell'_i) m'_i.$$

As an $\mathcal{A}_1$-module, P_n is generated by the $\Sigma^n q_0 \in \Sigma^n Ɛ$, $\Sigma^n e_0, \Sigma^n e_1 \in \Sigma^n E$, and $\Sigma^n 1 \in \Sigma^n \mathcal{A}_1$ of each summand. If w is one of these generators, we have

$$(\mathrm{id}_M \otimes p_n)\mu\theta w = U \otimes \lambda_n w,$$

where $p_n : M_h \to M_h/M_h^{[n-1]}$ is the quotient map. To see this, we can separate the degree n (i.e. $\Sigma^n q_0$, $\Sigma^n e_0$, and $\Sigma^n 1$) and $n+1$ (i.e. $\Sigma^n e_1$) cases and check that the m_i summands of μw (from Equation 7.2) are killed by p_n.

(i) If w has degree n, then each m_i has degree less than n and is killed by p_n.

(ii) If w has degree $n+1$, then because M vanishes in degree one [ABP67, Theorem 8.1], there are no terms in Equation 7.2 where m_i has degree n. So each m_i has degree less than n and is killed by p_n, as claimed.

Since we are working with $\mathcal{A}_1$-modules and $U \in M$ is annihilated by Sq^1 and Sq^2, the Cartan formula for the action of $\mathcal{A}_1$ on the tensor product implies that

$$(\mathrm{id}_M \otimes p_n)\mu\theta v = U \otimes \lambda_n v$$

for all $v \in P_n$ (rather than just for the generators).

Next, we want to show that there is a map $\mu_n : M_h/M_h^{[n-1]} \to M \otimes M_h/M_h^{[n-1]}$ such that the diagram

$$\begin{tikzcd} N \arrow[r, "\theta"] \arrow[d] & M_h \arrow[r, "\mu"] \arrow[d, "p_n"] & M \otimes M_h \arrow[d, "\mathrm{id}_M \otimes p_n"] \\ N/N^{[n-1]} \arrow[r, "\lambda_n"] & M_h/M_h^{[n-1]} \arrow[r, "\mu_n"] & M \otimes M_h/M_h^{[n-1]} \end{tikzcd} \tag{7.3}$$

commutes. To prove that such a μ_n exists, it suffices to show that $(\mathrm{id}_M \otimes p_n)\mu y = 0$ for each $y \in M_h^{[n-1]}$. To this end, let $y \in M_h^{[n-1]}$. Since $M_h^{[n-1]} = \theta(N^{[n-1]})$ (Notation 7.5), there exists $x \in N^{[n-1]}$ such that $y = \theta x$. Write $x = \sum_{i=1}^{\alpha} a_i x_i$, where $a_i \in \mathcal{A}$ and x_i are generators for the summands that constitute $N^{[n-1]}$. In particular, each x_i has degree less than or equal to $n-1$. By Equation 7.2, we have

$$\begin{aligned} \mu y &= \mu\theta\left(\sum_{i=1}^{\alpha} a_i x_i\right) \\ &= \sum_{i=1}^{\alpha} a_i \mu\theta x_i \\ &= \sum_{i=1}^{\alpha} a_i \left(U \otimes \theta x_i + \sum_{j=1}^{\beta_i} \ell_{i,j} \otimes m_{i,j}\right), \end{aligned}$$

where $m_{i,j}$ has degree less than or equal to $n-2$ (as $\deg(x_i) \le n-1$). Thus $m_{i,j} \in M_h^{[n-1]}$ for all i,j, so $(\mathrm{id}_M \otimes p_n)\mu y = 0$ and therefore the map μ_n exists and Diagram 7.3 commutes.

We are finally read to show that $\lambda_n|_{N^{[n]}/N^{[n-1]}}$ is injective, which we do by contradiction. Suppose that $v \in N^{[n]}/N^{[n-1]}$ is non-zero and satisfies $\lambda_n v = 0$. Let $\{v_i\}_{i\in I}$ be a homogeneous basis of P_n as a $\mathbb{Z}/2\mathbb{Z}$-vector space. Since $N^{[n]}/N^{[n-1]}$ is generated as an $\mathcal{A}$-module by P_n, we can write $v = \sum_{i\in I} a_i v_i$ for some homogeneous $a_i \in \mathcal{A}$.

If $a_i \in \mathcal{A}\mathrm{Sq}^1 + \mathcal{A}\mathrm{Sq}^2$, then we have $a_i = a_i'\mathrm{Sq}^1 + a_i''\mathrm{Sq}^2$ for some $a_i', a_i'' \in \mathcal{A}$. We can then write $a_i v_i = (a_i'\mathrm{Sq}^1 + a_i''\mathrm{Sq}^2)v_i = a_i' w_i' + a_i'' w_i''$, where $w_i' = \mathrm{Sq}^1 v_i$ and $w_i'' = \mathrm{Sq}^2 v_i$. Since $w_i', w_i'' \in P_n$, we can rewrite w_i' and w_i'' as linear combinations of $\{v_j\}_{j\in I}$. Since Sq^1 and Sq^2 increase degree, the basis elements in the linear combination for $a_i'w_i' + a_i''w_i''$ have greater degree than that of v_i. If a_i' or a_i'' is an element of $\mathcal{A}\mathrm{Sq}^1 + \mathcal{A}\mathrm{Sq}^2$, repeat this procedure. Since $\mathcal{J}$, E, $\mathcal{A}_1$, and hence P_n are all bounded above, this procedure eventually stabilizes. It follows that we can always write $v = \sum_{i \in I} a_i v_i$ with $a_i \notin \mathcal{A}\mathrm{Sq}^1 + \mathcal{A}\mathrm{Sq}^2$.

Let $k := \max_{i \in I}\{\deg(a_i)\}$, which exists since all but finitely many a_i must be zero. Let $i_1, \ldots, i_\alpha$ be the indices such that $\deg(a_{i_j}) = k$. By Diagram 7.3, our assumption $\lambda_n v = 0$ implies that $(\mathrm{id}_M \otimes p_n)\mu\theta v = 0$. (Here we conflate $v \in N^{[n]}/N^{[n-1]}$ with any choice of lift $v \in N^{[n]}$, since Diagram 7.3 commutes.) Then

$$\begin{aligned} 0 &= (\mathrm{id}_M \otimes p_n)\mu\theta v \\ &= (\mathrm{id}_M \otimes p_n)\mu\theta\left(\sum_{i \in I} a_i v_i\right) \\ &= \sum_{i \in I} a_i (\mathrm{id}_M \otimes p_n)\mu\theta v_i \\ &= \sum_{i \in I} a_i (U \otimes \lambda_n v_i) \\ &= \sum_{j=1}^{\alpha} a_{i_j} U \otimes \lambda_n v_{i_j} + x, \end{aligned} \tag{7.4}$$

where x is a sum of terms belonging to $M^\beta \otimes M_h/M_h^{[n-1]}$ for $\beta < k$. Recall that $\lambda_n|_{P_n}$ is injective (Lemma 7.7), so $\lambda_n v_{i_1}, \ldots, \lambda_n v_{i_\alpha}$ are linearly independent. It thus follows from Equation 7.4 that $a_{i_1} U = \ldots = a_{i_\alpha} U = 0$. But the submodule of M generated by U is isomorphic to $\mathcal{A}/(\mathcal{A}\mathrm{Sq}^1 + \mathcal{A}\mathrm{Sq}^2)$, and we chose $a_{i_j} \notin \mathcal{A}\mathrm{Sq}^1 + \mathcal{A}\mathrm{Sq}^2$, which yields the desired contradiction. Hence $v = 0$. $\square$

We are now ready to show that our extension $\theta : N \to M_h$ of $\bar{\theta} : \overline{N} \to M_h$ (Notation 7.3) is indeed an isomorphism. The general idea is to use Lemma 7.8 to inductively show that θ is injective. Paired with Lemma 7.4, we will find that θ is an isomorphism.

Proposition 7.9. *There exists a set of homogeneous elements $Z \subset M_h$ and an isomorphism $\theta : N \to M_h$ extending $\bar{\theta} : \overline{N} \to M_h$ along the inclusion $\overline{N} \to N$, where*

$$N = \overline{N} \oplus \bigoplus_{z \in Z} \Sigma^{\deg z} \mathcal{A}.$$

Proof. Let Z and θ be as in Setup 7.2 and Notation 7.3. We will induct on n, with our induction hypothesis the statement that $\theta_n : N^{[n]} \to M_h^{[n]}$ is an isomorphism. To simplify, our base case is $n = -1$, so that $N^{[n]}$ and $M_h^{[n]}$ are both trivial and there is nothing to check.

Now, assuming that θ_{n-1} is an isomorphism, it suffices to show that θ_n is injective by Lemma 7.4. To this end, consider the diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & N^{[n-1]} \arrow[r] \arrow[d, "\theta_{n-1}"] & N^{[n]} \arrow[r] \arrow[d, "\theta_n"] & N^{[n]}/N^{[n-1]} \arrow[d, "\lambda_n"] \\ 0 \arrow[r] & M_h^{[n-1]} \arrow[r] & M_h^{[n]} \arrow[r] & M_h^{[n]}/M_h^{[n-1]}. \end{tikzcd}$$

The rows are exact and $0 \to 0$ is an epimorphism. By the induction hypothesis θ_{n-1} is a monomorphism, and λ_n is a monomorphism by Lemma 7.8. The four lemma implies θ_n is a monomorphism. So θ_n is injective and hence an isomorphism.

So by induction, each θ_n is an isomorphism. If $v \in N$ is homogeneous, then $v \in N^{[n]}$ for some n. If $\theta v = 0$, then $\theta_n v = 0$, and therefore $v = 0$. Therefore θ is injective. It now follows from Lemma 7.4 that θ is an isomorphism. $\square$

The proof of the main theorem now follows formally.

Proof of Theorem 7.1. For each $z \in Z$, let $\mathrm{MSpin}^h \to \Sigma^{\deg z} H\mathbb{Z}/2\mathbb{Z}$ be the map classifying $z \in H^*\mathrm{MSpin}^h$. Together with the KSp-Pontryagin and elephant classes, we get a map

$$\mathrm{MSpin}^h \to \bigvee_{I \in \mathcal{P}_{\text{even}}} \mathrm{ksp}\langle 4|I|\rangle \vee \bigvee_{I \in \mathcal{P}_{\text{odd}}} \Sigma^{4|I|} F \vee \bigvee_{z \in Z} \Sigma^{\deg z} H\mathbb{Z}/2\mathbb{Z} \tag{7.5}$$

inducing θ in cohomology. Since $H^*\mathrm{MSpin}^h$ is finitely generated in each degree (by Proposition 4.1 and the Thom isomorphism), we can dualize to see that Equation 7.5 induces an isomorphism in homology with coefficients in $\mathbb{Z}/2\mathbb{Z}$. Hence this map is a 2-complete equivalence. Since MSpin^h and our wedge sum both have finitely generated homotopy groups in each degree (by Proposition 2.8, Lemma 5.3, and Bott periodicity), Equation 7.5 is a 2-local equivalence. $\square$

8. Calculating Spin^h cobordism groups

According to Milnor, calculating Spin cobordism groups is a "formidable computation" [Mil63, p. 202]. The Spin, Spin^c, and Spin^h cobordism groups are all 2-primary, so their splitting at $p = 2$ is sufficient to compute these groups. The formidable computation arises from two calculations: (i) the combinatorics of partitions that characterize the Anderson–Brown–Peterson splitting in the real and complex cases [ABP67] and Theorem 7.1 in the quaternionic case, and (ii) counting the Eilenberg–Mac Lane summands. We provide code at [Buc23] that performs these manipulations for us, as well as tables of $\pi_*\mathrm{MSpin}$ (Table 2), $\pi_*\mathrm{MSpin}^c$ (Table 3), and $\pi_*\mathrm{MSpin}^h$ (Table 4) for $0 \le * \le 99$. Tables for $0 \le * \le 19999$ are also available at [Buc23].

Remark 8.1. A table of $\pi_*\mathrm{MSpin}$ for $0 \le * \le 127$ (with an extra column recording additional information about the torsion) appears in [BN14, Section 10]. Nevertheless, we include Table 2 for the reader's convenience. A table of $\pi_*\mathrm{MSpin}^c$ for $0 \le * \le 59$ is given in [BG87, p. 5]. Values of $\pi_*\mathrm{MSpin}^h$ are given for $0 \leq * \leq 6$ in [Hu22, §3.5] and for $0 \leq * \leq 30$ in [Mil23, §4].

8.1. **Computing rank and torsion.** We used code to generate Tables 2, 3, and 4. In this section, we will explain the math behind this code.

8.1.1. *Rank.* Theorems 3.5, 3.6, and 7.1 tell us that the ranks of $\pi_*\mathrm{MSpin}$, $\pi_*\mathrm{MSpin}^c$, and $\pi_*\mathrm{MSpin}^h$ are determined by the combinatorics of partitions and the homotopy groups of various connective covers of KO, KU, and KSp, respectively. Putting this all together, we can derive formulas for the ranks of these bordism groups.

Notation 8.2. Let $p(i) = |\mathcal{P}(i)|$ and $p_1(i) = |\mathcal{P}_1(i)|$ denote the number of partitions of i and the number of partitions of i not containing 1, respectively.

Lemma 8.3. *We have*

$$\operatorname{rank} \pi_n\mathrm{MSpin} = \begin{cases} p(m) & n = 4m \geq 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. By Theorem 3.5 and Bott periodicity for ko (see Table 1), we find that

$$\operatorname{rank} \pi_{8m}\mathrm{MSpin} = \sum_{i=0}^{2m} p_1(i), \tag{8.1}$$

$$\operatorname{rank} \pi_{8m+4}\mathrm{MSpin} = \sum_{i=0}^{2m+1} p_1(i).$$

Partitions of i containing 1 are sums of the form $1 + s$ for s a partition of $i - 1$, so we have $p(i) = p_1(i) + p(i-1)$. That is, $p_1(i) = p(i) - p(i-1)$. By expanding the sums in Equation 8.1 in terms of $p(i)$, we have

$$\begin{aligned} \sum_{i=0}^{k} p_1(i) &= p_1(0) + \sum_{i=1}^{k}(p(i) - p(i-1)) \\ &= p(k) - p(0) + p_1(0) \\ &= p(k). \end{aligned}$$

Thus $\operatorname{rank} \pi_{8m}\mathrm{MSpin} = p(2m)$ and $\operatorname{rank} \pi_{8m+4}\mathrm{MSpin} = p(2m+1)$, or more simply

$$\operatorname{rank} \pi_{4m}\mathrm{MSpin} = p(m).$$

Since the free part of $\pi_*\mathrm{KO}$ is concentrated in degrees $4m \geq 0$, it follows that $\pi_*\mathrm{MSpin}$ is torsion in all other degrees. $\square$

Lemma 8.4. *We have*

$$\operatorname{rank} \pi_n\mathrm{MSpin}^c = \begin{cases} \sum_{i=0}^{m} p(i) & n = 4m \geq 0, \\ \sum_{i=0}^{m} p(i) & n = 4m + 2 \geq 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Recall that the free part of $\pi_*\mathrm{ku}\langle i\rangle$ is concentrated in degrees $2j \geq i$, and that each non-trivial free summand has rank 1. Thus by Theorem 3.6, the rank of $\pi_{4m}\mathrm{MSpin}^c$ is given by the sum $\sum_{i=0}^m p(i)$. The same argument holds for $\operatorname{rank}\pi_{4m+2}\mathrm{MSpin}^c$, as the connective covers in Theorem 3.6 proceed in multiples of 4. □

Lemma 8.5. *We have*

$$\operatorname{rank}\pi_n\mathrm{MSpin}^h = \begin{cases} \sum_{i=0}^m p(i) & n = 4m \geq 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The free part of $\pi_*\mathrm{ksp}\langle i\rangle$ is concentrated in degrees $4j \geq i$, and each non-trivial free summand has rank 1. The same is true of the spectra $\Sigma^{8k+4}F$, since $\pi_*\mathrm{ksp} \cong \pi_*F$ (Lemma 5.3). Theorem 7.1 thus implies that the rank of $\pi_{4m}\mathrm{MSpin}^h$ is given by the sum

$$\sum_{i=0}^{\lfloor m/2\rfloor} p(2i) + \sum_{i=0}^{\lceil m/2\rceil - 1} p(2i+1) = \sum_{i=0}^{m} p(i),$$

as desired. □

Note that we have just shown that $\operatorname{rank}\pi_{4n}\mathrm{MSpin}^h = \operatorname{rank}\pi_{4n}\mathrm{MSpin}^c$.

Corollary 8.6. *We have* $\operatorname{rank}\pi_{4n}\mathrm{MSpin}^h = \operatorname{rank}\pi_{4n}\mathrm{MSpin}^c = \operatorname{rank}\pi_{4n+2}\mathrm{MSpin}^c$ *for all* n.

Proof. This follows directly from Lemmas 8.4 and 8.5. □

8.1.2. *Torsion.* Besides the partition numbers, one needs to count the Eilenberg–Mac Lane summands in order to determine these groups. To do this, we can use Hilbert–Poincaré series representing the dimension of various $\mathcal{A}$-modules in each degree. If M is an $\mathcal{A}$-module, let $P(M)$ denote its Hilbert–Poincaré series.

Proposition 8.7. *We have the following Hilbert–Poincaré series:*

$$\begin{aligned} P(H^*\mathrm{MSpin}^h) &= \prod_{n\geq 2}(1-t^n)^{-1} \cdot \prod_{r\geq 2}(1-t^{2^r+1}), \\ P(\mathcal{A}) &= \prod_{n\geq 1}(1-t^{2^n-1})^{-1}, \\ P(H^*\mathrm{ksp}\langle 8k\rangle) &= \prod_{n\geq 3}(1-t^{2^n-1})^{-1} \cdot \frac{t^{8k}(1+t^2+t^3)}{(1-t^4)(1-t^6)}, \\ P(H^*\Sigma^{8k+4}F) &= \prod_{n\geq 3}(1-t^{2^n-1})^{-1} \cdot \frac{t^{8k+4}(1+t+2t^2+t^3+t^4+t^5)}{(1-t^4)(1-t^6)}. \end{aligned}$$

Proof. For $P(H^*\mathrm{MSpin}^h) = P(H^*\mathrm{BSpin}^h)$, recall that the cohomology of BSpin^h is a polynomial ring, so its Hilbert–Poincaré series is a product with a factor of $(1-t^n)^{-1}$ for each generator of degree n. There is a generator in degrees $i \geq 2$ such that $i \neq 2^{k+2}+1$ (for $k \geq 0$) by Proposition 4.3.

The series for $\mathcal{A}$ is given in [ABP66, Theorem 1.11].

Since $H^*\mathrm{ksp}\langle 8k\rangle \cong \Sigma^{8k} H^*\mathrm{ksp} \cong \Sigma^{8k} \text{¿}_{\mathcal{A}}$, we can use the exact sequence

$$0 \longrightarrow \Sigma^3 \text{¿}_{\mathcal{A}} \longrightarrow \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{A}\mathrm{Sq}^3 \longrightarrow 0,$$

which implies $t^3 P(\text{¿}_{\mathcal{A}}) = P(\mathcal{A}) - P(\mathcal{A}/\mathcal{A}\mathrm{Sq}^3)$. From [ABP67, Theorem 1.11], we know

$$P(\mathcal{A}/\mathcal{A}\mathrm{Sq}^3) = \prod_{n\geq 3}(1-t^{2^n-1})^{-1} \cdot (1-t^4)^{-1}(1-t^6)^{-1}(1+t+t^2+t^3+t^4).$$

It follows that

$$\begin{aligned} P(H^*\mathrm{ksp}\langle 8k\rangle) &= t^{8k} P(\text{¿}_{\mathcal{A}}) \\ &= \frac{t^{8k}}{t^3}\prod_{n\geq 3}(1-t^{2^n-1})^{-1}\cdot\left(\frac{1}{(1-t)(1-t^3)} - \frac{1+t+t^2+t^3+t^4}{(1-t^4)(1-t^6)}\right) \\ &= \prod_{n\geq 3}(1-t^{2^n-1})^{-1}\cdot\frac{t^{8k}(1+t^2+t^3)}{(1-t^4)(1-t^6)}. \end{aligned}$$

Finally, for $E_{\mathcal{A}}$ we use the exact sequence

$$0 \longrightarrow \Sigma E_{\mathcal{A}} \longrightarrow \mathcal{A} \longrightarrow \mathcal{A}/(\mathcal{A}\mathrm{Sq}^1 + \mathcal{A}\mathrm{Sq}^2) \longrightarrow 0$$

to get the equation $tP(E_{\mathcal{A}}) = P(\mathcal{A}) - P(\mathcal{A}/(\mathcal{A}\mathrm{Sq}^1 + \mathcal{A}\mathrm{Sq}^2))$. From [ABP66, Theorem 1.11], we have

$$P(\mathcal{A}/(\mathcal{A}\mathrm{Sq}^1 + \mathcal{A}\mathrm{Sq}^2)) = \prod_{n\geq 3}(1-t^{2^n-1})^{-1}\cdot(1-t^4)^{-1}(1-t^6)^{-1}.$$

Therefore

$$\begin{aligned} P(E_{\mathcal{A}}) &= \frac{1}{t}\prod_{n\geq 3}(1-t^{2^n-1})^{-1}\cdot\big((1-t)^{-1}(1-t^3)^{-1} - (1-t^4)^{-1}(1-t^6)^{-1}\big) \\ &= \prod_{n\geq 3}(1-t^{2^n-1})^{-1}\cdot\frac{1+t+2t^2+t^3+t^4+t^5}{(1-t^4)(1-t^6)}. \end{aligned}$$

□

We can now describe the generating function for the number of $H\mathbb{Z}/2\mathbb{Z}$ summands in each degree.

Corollary 8.8. *Let $R(t) := \sum_{k\geq 0} r_k t^k$, where r_k is the number of $\Sigma^k H\mathbb{Z}/2\mathbb{Z}$ summands of* MSpin^h. *Then*

$$\begin{aligned} R(t) = (1-t)\prod_{\substack{n\geq 2 \\ n\neq 2^r\pm 1}}(1-t^n)^{-1} - \frac{1}{(1+t)(1+t^2)(1+t^3)}\sum_{k\geq 0} t^{8k}\Big(&p(2k)(1+t^2+t^3) \\ &+ p(2k+1)(t^4+t^5+2t^6+t^7+t^8+t^9)\Big). \end{aligned}$$

Proof. Theorem 7.1 implies that

$$P(H^*\mathrm{MSpin}^h) = \sum_{k\geq 0}\left(\sum_{\mathcal{P}(2k)} P(H^*\mathrm{ksp}\langle 8k\rangle) + \sum_{\mathcal{P}(2k+1)} P(H^*\Sigma^{8k+4}F)\right) + R\cdot P(\mathcal{A}).$$

Solving for R, we obtain

$$\begin{aligned} R(t) = &\prod_{n\geq 2}(1-t^n)^{-1}\cdot\prod_{r\geq 2}(1-t^{2^r+1})\cdot\prod_{r\geq 1}(1-t^{2^r-1}) \\ &-\prod_{r=1}^{2}(1-t^{2^r-1})\sum_{k\geq 0}\left(\frac{t^{8k}}{(1-t^4)(1-t^6)}\left(\sum_{\mathcal{P}(2k)}(1+t^2+t^3)\right.\right. \\ &\left.\left.+\sum_{\mathcal{P}(2k+1)} t^3((1+t)(1+t^2)(1+t^3)-1)\right)\right). \end{aligned}$$

The result follows from simplifying this expression. □

To give the generating function for the torsion part of $\pi_*\mathrm{MSpin}^h$, it remains to add the torsion contributions from the $\mathrm{ksp}\langle 4(2k)\rangle$ and $\Sigma^{4(2k+1)}F$ summands. Bott periodicity for KSp and Lemma 5.3 give us the torsion, which we restate here for convenience.

Lemma 8.9. *Let $k \geq 0$. Then*

$$(\pi_*\mathrm{ksp}\langle 8k\rangle)_{\mathrm{tors}} \cong (\pi_*\Sigma^{8k+4}F)_{\mathrm{tors}} \cong \begin{cases} \mathbb{Z}/2 & * = 8n+5 \text{ with } n\geq k, \\ \mathbb{Z}/2 & * = 8n+6 \text{ with } n \geq k, \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 8.10. *Let $R(t)$ be the generating series given in Corollary 8.8. Let $S(t) := \sum_{k\geq 0} s_k t^k$, where $(\pi_k\mathrm{MSpin}^h)_{\mathrm{tors}} \cong (\mathbb{Z}/2)^{s_k}$. Then*

$$S(t) = R(t) + (t^5+t^6)\sum_{k\geq 0} t^{8k}(p(2k)+p(2k+1)).$$

Proof. It suffices to show that the generating series for the torsion groups coming from $\mathrm{ksp}\langle 8k\rangle$ and $\Sigma^{8k+4}F$ is $(t^5+t^6)\sum_{k\geq 0} t^{8k}(p(2k)+p(2k+1))$. By Theorem 7.1, the $\mathrm{ksp}\langle 8k\rangle$ terms are indexed over $\mathcal{P}(2k)$, while the $\Sigma^{8k+4}F$ terms are indexed over $\mathcal{P}(2k+1)$. By Lemma 8.9, the coefficients $p(2k)$ and $p(2k+1)$ are each weighted by $t^{8k}(t^5+t^6)$. □

8.2. **Growth rates.** Since Spin, Spin^c, and Spin^h bordism groups are combinatorially determined, we can analyze their growth combinatorially as well. The asymptotic growth of partitions is due to Hardy and Ramanujan [HR00b, HR00a], which allows us to prove the asymptotic growth of the ranks of these bordism groups.

Proposition 8.11. *Let $r_n M := \operatorname{rank} \pi_n M$, where M is any spectrum. Then*

$$\frac{\exp(\pi\sqrt{2n/3})}{4n\sqrt{3}} \sim r_{4n}\mathrm{MSpin},$$
$$\frac{\exp(\pi\sqrt{2n/3})}{2\pi\sqrt{2n}} \sim r_{4n}\mathrm{MSpin}^c = r_{4n+2}\mathrm{MSpin}^c = r_{4n}\mathrm{MSpin}^h.$$

Proof. Hardy–Ramanujan [HR00b, HR00a] proved that

$$p(i) \sim \frac{\exp(\pi\sqrt{2i/3})}{4i\sqrt{3}}.$$

Thus $r_{4n}\mathrm{MSpin} \sim \frac{\exp(\pi\sqrt{2n/3})}{4n\sqrt{3}}$ by Lemma 8.3. We proved the equality $\operatorname{rank} \pi_{4n}\mathrm{MSpin}^c = \operatorname{rank} \pi_{4n}\mathrm{MSpin}^h = \sum_{i=0}^n p(i)$ in Corollary 8.6. It thus suffices to calculate

$$\sum_{i=0}^n p(i) \sim \frac{\exp(\pi\sqrt{2n/3})}{2\pi\sqrt{2n}},$$

which follows from Hardy–Ramanujan's asymptotic formula for $p(n)$, Gupta's formula [Gup46]

$$\sum_{i=0}^{n-1} p(i) \sim \frac{p(n)\sqrt{6n}}{\pi},$$

and the calculation $\lim_{n\to\infty} \dfrac{p(n+1)\sqrt{n+1}}{p(n)\sqrt{n}} = 1$. □

Remark 8.12. The asymptotic growth of partitions (and hence the ranks of $\pi_*\mathrm{MSpin}$, $\pi_*\mathrm{MSpin}^c$, and $\pi_*\mathrm{MSpin}^h$) are calculated using the circle method. This same method could be used to calculate the growth of the torsion parts as well. For example, the growth of the torsion part of $\pi_*\mathrm{MSpin}^h$ is given by the growth of the coefficients of $S(t)$, whose poles all lie on the unit circle. We will not investigate the asymptotics of the torsion parts of $\pi_*\mathrm{MSpin}^h$ here.

9. Characterizing Spin^h cobordism classes

In Section 8, we saw that we can explicitly compute Spin^h bordism groups up to any desired degree (contingent upon having enough computational power). However, these computations only describe the Spin^h bordism groups abstractly. What we really want from $\pi_*\mathrm{MSpin}^h$ is an understanding of the geometry of Spin^h manifolds up to cobordism.

Theorem 1.1 implies that the KSp-characteristic classes given in Definition 5.17, together with $H\mathbb{Z}/2\mathbb{Z}$-characteristic classes, can be used to distinguish Spin^h cobordism classes (by evaluating on an appropriate homology class). In this section, we will show that instead of using the elephant classes for odd partitions, it suffices to use KSp-Pontryagin classes for all partitions.

Setup 9.1. Recall that a Spin^h manifold is a smooth compact manifold without boundary, equipped with a Spin^h structure on its stable normal bundle ν. If M is a smooth compact n-manifold without boundary, the Pontryagin–Thom construction gives a map of spectra $\theta : \Sigma^n\mathbb{S} \to \mathrm{Th}(\nu)$, where $\mathrm{Th}(\nu)$ is the Thom spectrum of the stable normal bundle of M.

The unit map $\mathbb{S} \to \mathrm{KO} \cong \mathbb{S} \wedge \mathrm{KO}$ induces a KO-homology class $1 \in \mathrm{KO}_0\mathbb{S}$. Shifting and then transferring along θ gives us a class $\theta_*1 \in \mathrm{KO}_n \mathrm{Th}(\nu)$. We also have a class $a \in \mathrm{KSp}^0 \mathrm{Th}(\nu)$ given by the composition of $h\upsilon : \mathrm{Th}(\nu) \to \mathrm{MSpin}^h$ (coming from the Spin^h structure on M) and the Atiyah–Bott–Shapiro map $\varphi^h : \mathrm{MSpin}^h \to \mathrm{KSp}$ (Proposition 2.9). The KSp-homology class $\theta_*1 \frown a$ can be thought of as a sort of KSp-fundamental class of M as a Spin^h manifold.

Definition 9.2. Let I be a partition and M a Spin^h-manifold. The I^{th} KSp- *characteristic number* of M is $\langle \pi_h^I(\nu), \theta_*1 \frown a\rangle \in \mathrm{KSp}_n$, where $\pi_h^I \in \mathrm{KO}^0(\mathrm{BSpin}^h)$ is the KO-Pontryagin class.

Diagramatically, the KSp-homology class $\theta_*1 \frown a$ on M is given by

$$\begin{tikzcd}
\mathrm{Th}(\nu)\wedge\mathrm{KO} \arrow[r, "\delta\wedge\mathrm{id}"] & M_+\wedge\mathrm{Th}(\nu)\wedge\mathrm{KO} \arrow[r, "\mathrm{id}\wedge a\wedge\mathrm{id}"] & M_+\wedge\mathrm{KSp}\wedge\mathrm{KO} \arrow[d, "\mathrm{id}\wedge\mu"] \\
\Sigma^n\mathbb{S} \arrow[u, "\theta_*1"] & & M_+\wedge\mathrm{KSp},
\end{tikzcd}$$

where δ is the Thom diagonal and $\mu : \mathrm{KSp}\wedge\mathrm{KO} \to \mathrm{KSp}$ is the KO-module structure. The I^{th} KSp-characteristic number of M is then given by

$$\Sigma^n\mathbb{S} \xrightarrow{\theta_*1\frown a} M_+\wedge\mathrm{KSp} \xrightarrow{\pi_h^I(\nu)\wedge\mathrm{id}} \mathrm{KO}\wedge\mathrm{KSp} \xrightarrow{\mu} \mathrm{KSp}.$$

The main lemma of this section is that KSp-characteristic numbers are indeed related to our KSp-characteristic classes.

Lemma 9.3. *If M is a* Spin^h*-manifold, then the I^{th}* KSp-*characteristic number can be computed as the composite*

$$\Sigma^n\mathbb{S} \xrightarrow{\theta} \mathrm{Th}(\nu) \xrightarrow{h\upsilon} \mathrm{MSpin}^h \xrightarrow{\kappa^I} \mathrm{KSp},$$

where κ^I is the I^{th} KSp-*Pontryagin class (see Remark 5.18).*

Proof. As before, let $\delta : \mathrm{Th}(\nu) \to M_+\wedge\mathrm{Th}(\nu)$ denote the Thom diagonal. Let $e : \mathbb{S} \to \mathrm{KO}$ denote the unit map and $\mu : \mathrm{KSp}\wedge\mathrm{KO} \to \mathrm{KSp}$ the KO-module multiplication of KSp.

The diagram

$$\begin{tikzcd}
\Sigma^n\mathbb{S} \wedge \mathrm{KO} \arrow[r, "\theta\wedge\mathrm{id}"] & \mathrm{Th}(\nu)\wedge\mathrm{KO} \arrow[r, "\delta\wedge\mathrm{id}"] & M_+ \wedge \mathrm{Th}(\nu)\wedge\mathrm{KO} \\
\Sigma^n\mathbb{S} \wedge \mathbb{S} \arrow[u, "\mathrm{id}\wedge e"] \arrow[r, "\theta\wedge\mathrm{id}"] & \mathrm{Th}(\nu)\wedge\mathbb{S} \arrow[u, "\mathrm{id}\wedge e"] \arrow[r, "\delta\wedge\mathrm{id}"] & M_+ \wedge \mathrm{Th}(\nu)\wedge\mathbb{S} \arrow[u, "\mathrm{id}\wedge\mathrm{id}\wedge e"] \\
\Sigma^n\mathbb{S} \arrow[u, "\rho"] \arrow[r, "\theta"] & \mathrm{Th}(\nu) \arrow[u, "\rho"] \arrow[r, "\delta"] & M_+ \wedge \mathrm{Th}(\nu) \arrow[u, "\rho"]
\end{tikzcd} \tag{9.1}$$

commutes, because the unit isomorphisms in a symmetric monoidal category are natural. Next, the diagram

$$\begin{tikzcd}
\mathrm{Th}(\nu) \arrow[r, "\delta"] \arrow[d, "h\upsilon"] & M_+ \wedge \mathrm{Th}(\nu) \arrow[r, "\pi_h^I(\nu)\wedge a"] \arrow[d, "\nu\wedge h\upsilon"] & \mathrm{KO}\wedge\mathrm{KSp} \arrow[r, "\mu"] \arrow[d, "\mathrm{id}"] & \mathrm{KSp} \arrow[d, "\mathrm{id}"] \\
\mathrm{MSpin}^h \arrow[r, "\delta"] & \mathrm{BSpin}^h \wedge \mathrm{MSpin}^h \arrow[r, "\pi_h^I\wedge\varphi^h"] & \mathrm{KO}\wedge\mathrm{KSp} \arrow[r, "\mu"] & \mathrm{KSp}
\end{tikzcd} \tag{9.2}$$

commutes by naturality of the Thom diagonal. Finally, consider the diagram

$$\begin{tikzcd}
M_+ \wedge \mathrm{Th}(\nu)\wedge\mathrm{KO} \arrow[r, "\mathrm{id}\wedge a\wedge\mathrm{id}"] & M_+ \wedge \mathrm{KSp}\wedge\mathrm{KO} \arrow[d, "\mathrm{id}\wedge\mu"] \\
M_+ \wedge \mathrm{Th}(\nu)\wedge\mathbb{S} \arrow[u, "\mathrm{id}\wedge\mathrm{id}\wedge e"] & M_+ \wedge \mathrm{KSp} \arrow[d, "\pi_h^I(\nu)\wedge\mathrm{id}"] \\
M_+ \wedge \mathrm{Th}(\nu) \arrow[u, "\rho"] \arrow[r, "\pi_h^I(\nu)\wedge a"] & \mathrm{KO}\wedge\mathrm{KSp}.
\end{tikzcd} \tag{9.3}$$

To see that Diagram 9.3 commutes, we use the identity axiom for KSp as a KO-module, which states that the diagram

$$\begin{tikzcd}
\mathrm{KSp}\wedge\mathbb{S} \arrow[r, "\mathrm{id}\wedge e"] & \mathrm{KSp}\wedge\mathrm{KO} \arrow[d, "\mu"] \\
\mathrm{KSp} \arrow[u, "\rho"] \arrow[r, "\mathrm{id}"] & \mathrm{KSp}
\end{tikzcd}$$

commutes. Therefore the diagram

$$\begin{tikzcd}
\mathrm{Th}(\nu)\wedge\mathbb{S} \arrow[r, "a\wedge\mathrm{id}"] & \mathrm{KSp}\wedge\mathbb{S} \arrow[r, "\mathrm{id}\wedge e"] & \mathrm{KSp}\wedge\mathrm{KO} \arrow[d, "\mu"] \\
\mathrm{Th}(\nu) \arrow[u, "\rho"] \arrow[r, "a"] & \mathrm{KSp} \arrow[u, "\rho"] \arrow[r, "\mathrm{id}"] & \mathrm{KSp}
\end{tikzcd}$$

commutes. Since $(\mathrm{id}\wedge e)\circ(a\wedge\mathrm{id}) = (a\wedge\mathrm{id})\circ(\mathrm{id}\wedge e)$, the diagram

$$\begin{tikzcd}
\mathrm{Th}(\nu)\wedge\mathbb{S} \arrow[r, "\mathrm{id}\wedge e"] & \mathrm{Th}(\nu)\wedge\mathrm{KO} \arrow[r, "a\wedge\mathrm{id}"] & \mathrm{KSp}\wedge\mathrm{KO} \arrow[d, "\mu"] \\
\mathrm{Th}(\nu) \arrow[u, "\rho"] \arrow[r, "a"] & \mathrm{KSp} \arrow[r, "\mathrm{id}"] & \mathrm{KSp}
\end{tikzcd} \tag{9.4}$$

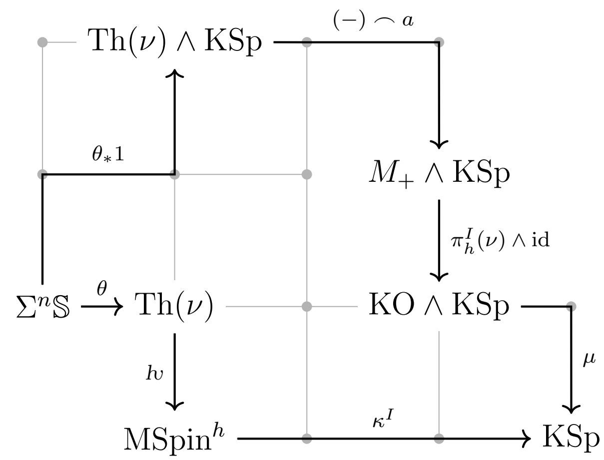


FIGURE 5. Two ways of computing KSp-characteristic numbers

commutes. Smashing Diagram 9.4 on the left with M_+, we find that

(9.5)
$$\begin{tikzcd}
M_+ \wedge \mathrm{Th}(\nu) \wedge \mathrm{KO} \arrow[rr, "\mathrm{id}\wedge a\wedge\mathrm{id}"] & & M_+ \wedge \mathrm{KSp} \wedge \mathrm{KO} \arrow[d, "\mathrm{id}\wedge\mu"] \\
M_+ \wedge \mathrm{Th}(\nu) \wedge \mathbb{S} \arrow[u, "\mathrm{id}\wedge\mathrm{id}\wedge e"] & & M_+ \wedge \mathrm{KSp} \\
M_+ \wedge \mathrm{Th}(\nu) \arrow[u, "\rho"] \arrow[urr, "\mathrm{id}\wedge a"] & &
\end{tikzcd}$$

commutes. To complete the commutativity of Diagram 9.3, we observe that

$$\begin{tikzcd}
 & M_+ \wedge \mathrm{KSp} \arrow[d, "\pi_h^I(\nu)\wedge\mathrm{id}"] \\
M_+ \wedge \mathrm{Th}(\nu) \arrow[ur, "\mathrm{id}\wedge a"] \arrow[r, "\pi_h^I(\nu)\wedge a"] & \mathrm{KO} \wedge \mathrm{KSp}
\end{tikzcd}$$

commutes. To conclude the lemma, we stitch together Diagrams 9.1, 9.2, and 9.3 and take two different routes $\Sigma^n\mathbb{S} \to \mathrm{KSp}$. For the reader's convenience, we depict these routes in Figure 5. $\square$

We are now ready to prove Theorem 1.2, which is an analog of [Wal60, Corollary 1].

Theorem 9.4. *Two* Spin^h*-manifolds are* Spin^h*-cobordant if and only if their* KSp- *and* $\mathbb{Z}/2\mathbb{Z}$*-characteristic numbers agree.*

Proof. First, we form the composition

$$\begin{tikzcd} & \bigvee_{I\in\mathcal{P}_{\mathrm{even}}}\mathrm{ksp}\langle 4|I|\rangle \vee \bigvee_{I\in\mathcal{P}_{\mathrm{odd}}}\Sigma^{4|I|}F \vee \bigvee_{z\in Z}\Sigma^{\deg z}H\mathbb{Z}/2\mathbb{Z} \arrow[d, "\psi"] \\ \mathrm{MSpin}^h \arrow[ur, bend left, "\phi"] \arrow[r, "\psi\circ\phi"] & \bigvee_{I\in\mathcal{P}}\mathrm{ksp}\langle 4|I|\rangle \vee \bigvee_{z\in Z}\Sigma^{\deg z}H\mathbb{Z}/2\mathbb{Z}, \end{tikzcd}$$

where ψ is given by the identity maps on the $\mathrm{ksp}\langle 4|I|\rangle$ (when $I \in \mathcal{P}_{\mathrm{even}}$) and $\Sigma^{\deg z}H\mathbb{Z}/2\mathbb{Z}$ summands, and by the map $\Sigma^{4|I|}F \to \mathrm{ksp}\langle 4|I|\rangle$ for $I \in \mathcal{P}_{\mathrm{odd}}$. The map ϕ is the splitting of Theorem 1.1, so ϕ is a 2-local equivalence. Taking homotopy groups, we find that

$$\ker\left((\psi\circ\phi)_* : \pi_*\mathrm{MSpin}^h \to \bigoplus_{I\in\mathcal{P}}\pi_*\mathrm{ksp}\langle 4|I|\rangle \oplus \bigoplus_{z\in Z}\pi_*\Sigma^{\deg z}H\mathbb{Z}/2\mathbb{Z}\right)$$

is trivial. To see this, note that ϕ induces an isomorphism (in particular, an injection) on homotopy groups. Similarly, ψ induces an injection on homotopy groups, since ψ_* is a direct sum of identity maps and copies of the inclusion $2\mathbb{Z} \to \mathbb{Z}$. Thus $(\psi\circ\phi)_* = \psi_*\circ\phi_*$ is an injection.

The previous paragraph suggests that $\psi\circ\phi$ can separate Spin^h-cobordism classes. Indeed, two Spin^h manifolds M_1 and M_2 are Spin^h-cobordant if and only if the class of $M = M_1 - M_2$ corresponds to $0 \in \pi_*\mathrm{MSpin}^h$ (under Pontryagin–Thom). Since $(\psi\circ\phi)_*$ is injective, $[M]$ corresponds to $0 \in \pi_*\mathrm{MSpin}^h$ if and only if $[M]$ maps to zero in each $\pi_*\mathrm{ksp}\langle 4|I|\rangle$ and each $\pi_*\Sigma^{\deg z}H\mathbb{Z}/2\mathbb{Z}$.

It remains to show that $(\psi\circ\phi)_*$ is the direct sum of the KSp- and $\mathbb{Z}/2\mathbb{Z}$-characteristic numbers. If I is a partition, then the element in $\pi_*\mathrm{ksp}\langle 4|I|\rangle$ determined by $[M]$ is

$$\Sigma^n\mathbb{S} \longrightarrow \mathrm{Th}(\nu) \longrightarrow \mathrm{MSpin}^h \longrightarrow \mathrm{ksp}\langle 4|I|\rangle,$$

which is precisely the I^{th} KSp-characteristic number of M by Lemma 9.3. Similarly, for $z \in Z$, the element of $\pi_*\Sigma^{\deg z}H\mathbb{Z}/2\mathbb{Z}$ corresponding to $[M]$ is the sum of various ordinary $\mathbb{Z}/2\mathbb{Z}$-characteristic numbers of M arising from the expression of z in the polynomial basis of the Stiefel–Whitney classes. So if all the KSp- and $\mathbb{Z}/2\mathbb{Z}$-characteristic numbers of M_1 and M_2 agree, then they vanish for M, and therefore the element of $\pi_*\mathrm{MSpin}^h$ determined by M is zero.

Conversely, if M_1 and M_2 are Spin^h-cobordant, then the KSp-characteristic numbers of M all vanish. Moreover, M_1 and M_2 being Spin^h-cobordant implies that their underlying unoriented manifolds are cobordant, and two unoriented manifolds are cobordant if and only if their Stiefel–Whitney numbers agree [Tho54]. It follows that the Stiefel–Whitney numbers of M are all zero as well. □

Remark 9.5. Theorem 9.4 can be summarized by saying that two Spin^h manifolds are Spin^h-cobordant if and only if their underlying unoriented manifolds are cobordant and all of their KSp-characteristic numbers agree.

10. Potential applications

In this section, we list a few more problems of interest in Spin^h cobordism theory.

10.1. **Explicit representatives of generators.** As seen in Section 8, we can now calculate the bordism groups $\Omega_*^{\mathrm{Spin}^h}$ in any degree (within the bounds of time and computational power). It would be desirable to have explicit Spin^h manifolds whose classes are generators in $\Omega_*^{\mathrm{Spin}^h}$.

Problem 10.1. Write $\pi_n \mathrm{MSpin}^h \cong \mathbb{Z}^{r_n} \times (\mathbb{Z}/2\mathbb{Z})^{t_n}$. Given a dimension n, find n-dimensional Spin^h manifolds $M_1, \dots, M_{r_n}, N_1, \dots, N_{t_n}$ such that $[M_1], \dots, [N_{t_n}]$ generate $\Omega_n^{\mathrm{Spin}^h}$.

Example 10.2. Since $\pi_n \mathrm{MSpin}^h$ is trivial for $n \in \{1, 2, 3, 7, 11\}$, Problem 10.1 is trivial in these dimensions. We can also make a few remarks in some small non-trivial dimensions.

(i) In dimension 0, $\pi_0 \mathrm{MSpin}^h \cong \mathbb{Z}$ is generated by a point with a choice of one of two Spin^h structures.

(ii) In dimension 4, one can use the Adams spectral sequence for the cofiber of the map $\mathrm{MSpin}^c \to \mathrm{MSpin}^h$ to show that the map $\pi_4 \mathrm{MSpin}^c \to \pi_4 \mathrm{MSpin}^h$ is injective. However, we do not know how to characterize this injection in terms of Spin^c and Spin^h manifolds.

(iii) In dimension 5, $\pi_5 \mathrm{MSpin}^h \cong (\mathbb{Z}/2\mathbb{Z})^2$ is generated by the Wu manifold $W = \mathrm{SU}(3)/\mathrm{SO}(3)$ and $S^1 \times S^4$ with a non-bounding Spin^h structure [Hu22, p. 37].

Recall that W admits a Spin^h-structure [AM21, Theorem 1.4], while W does not admit a Spin^c-structure [LM89, p. 393]. Moreover, $H^5(W; \mathbb{Z}/2\mathbb{Z})$ is generated by $w_2 w_3$ of the stable normal bundle [LM89, p. 393], so we are able to detect one of its KSp-characteristic numbers using ordinary cohomology.

The class $w_2 w_3 U_h \in H^* \mathrm{MSpin}^h$ comes from the lowest elephant class $\mathrm{MSpin}^h \to \Sigma^4 F$. Indeed, there are no $\Sigma^4 H\mathbb{Z}/2\mathbb{Z}$ summands in the splitting (see Table 4), and the only nonvanishing degree four cohomology class of ksp is $\mathrm{Sq}^4 y_0$, which maps to $w_4 U_h$. Because the Pontryagin–Thom map $\Sigma^5 \mathbb{S} \to \mathrm{Th}(\nu)$ maps the generator of $H^5 \Sigma^5 \mathbb{S}$ to $[W] \smile U_h$, the map $\Sigma^5 \mathbb{S} \to \mathrm{Th}(\nu)$ must send $w_2 w_3 U$ to the generator of $H^5 \Sigma^5 \mathbb{S}$ in cohomology. We can thus conclude that the map $\Sigma^5 \mathbb{S} \to \mathrm{KSp}$ is nontrivial, so the KSp-characteristic number of W determined by the partition (1) is 1.

This determines one of the components of $[W] \in \Omega_5^{\mathrm{Spin}^h} \cong (\mathbb{Z}/2\mathbb{Z})^2$. Determining the other component would likely require us to understand the K-theory of W.

(iv) In dimension 6, the Adams spectral sequence for the cofiber of $\mathrm{MSpin}^c \to \mathrm{MSpin}^h$ can be used to show that $\pi_6 \mathrm{MSpin}^c \to \pi_6 \mathrm{MSpin}^h$ is surjective. As in dimension 4, we do not know how to characterize this surjection in terms of Spin^c and Spin^h manifolds.

In private communication to the authors, Hu suggested $\mathrm{U}(3)/\mathrm{SO}(3)$ and $S^1 \times S^1 \times S^4$ as natural candidates for generators of $\pi_6\mathrm{MSpin}^h \cong (\mathbb{Z}/2\mathbb{Z})^2$. By Theorem 1.2, one could verify or refute this suggestion by computing the KSp-characteristic numbers of these two manifolds.

One question related to Problem 10.1 is about the relationship between free Spin^c and Spin^h cobordism classes.

Question 10.3. We saw in Corollary 8.6 that $\operatorname{rank} \pi_{4n}\mathrm{MSpin}^h = \operatorname{rank} \pi_{4n}\mathrm{MSpin}^c$. Is there a geometric explanation of this fact? In other words, is there a procedure for producing generators of the free part of $\Omega_{4n}^{\mathrm{MSpin}^h}$ from generators of the free part of $\Omega_{4n}^{\mathrm{MSpin}^c}$, and vice versa?

Remark 10.4. Question 10.3 is related to the injection $\pi_4\mathrm{MSpin}^c \to \pi_4\mathrm{MSpin}^h$ and surjection $\pi_6\mathrm{MSpin}^c \to \pi_6\mathrm{MSpin}^h$ coming from the Adams spectral sequence. Neither of these maps are isomorphisms, but they both have $(\mathbb{Z}/2\mathbb{Z})^2$ as their (co)kernel.

10.2. MSpin**-module structure of** MSpin^h**.** Since MSpin^h is an MSpin-module in the category of spectra, $\pi_*\mathrm{MSpin}^h$ is a $\pi_*\mathrm{MSpin}$-module in the category of rings. One can ask to characterize this module structure explicitly.

Problem 10.5. Calculate the module structure of $\pi_*\mathrm{MSpin}^h$ over the ring $\pi_*\mathrm{MSpin}$.

Problem 10.5 should be quite difficult, as even the ring structure of $\pi_*\mathrm{MSpin}$ is not completely understood [Lau03]. However, the ring structure of $\pi_*\mathrm{MSpin}$ is known modulo torsion [Sto66] (see also [ABP67, Theorem 2.8]). This suggests a suitable weakening of Problem 10.5.

Problem 10.6. Determine the structure of $\pi_*\mathrm{MSpin}^h/\text{torsion}$ as a module over the ring $\pi_*\mathrm{MSpin}/\text{torsion}$.

10.3. **Calculating Pin^h bordism groups.** Shortly after proving the 2-local splitting of MSpin, Anderson–Brown–Peterson computed the additive structure of $\Omega_*^{\mathrm{Pin}^-}$ using the isomorphism $\Omega_n^{\mathrm{Pin}^-} \cong \tilde{\Omega}_{n+1}^{\mathrm{Spin}}(\mathbb{RP}^\infty)$ [ABP69]. The additive structure of $\Omega_*^{\mathrm{Pin}^+}$ was computed by Kirby and Taylor [KT90].

The quaternionic pin groups $\mathrm{Pin}^{h\pm} := \mathrm{Pin}^\pm \times_{\{\pm 1\}} \mathrm{Sp}(1)$ were introduced by Freed and Hopkins under the notation $G^\pm$ [FH21, Proposition 9.16]. Using Theorem 7.1 as a starting point, computing the additive structure of $\Omega_*^{\mathrm{Pin}^{h\pm}}$ might be an accessible problem.

Problem 10.7. Compute the additive structure of $\Omega_*^{\mathrm{Pin}^{h\pm}}$.

For the Pin^{h-} case, one can try to construct a Smith isomorphism connecting Spin^h and Pin^{h-} cobordism.

Question 10.8. Is there is an isomorphism $\Omega_n^{\mathrm{Pin}^{h-}} \cong \tilde{\Omega}_{n+1}^{\mathrm{Spin}^h}(\mathbb{HP}^\infty)$ for each n?

Remark 10.9. A natural candidate for the morphism $\sigma : \tilde{\Omega}_{n+1}^{\mathrm{Spin}^h}(\mathbb{HP}^\infty) \to \Omega_n^{\mathrm{Pin}^{h-}}$ is as follows. Let M be a manifold representing a class in $\tilde{\Omega}_{n+1}^{\mathrm{Spin}^h}(\mathbb{HP}^\infty)$. Then there exists some $k \gg 0$ and a classifying map $f : M \to \mathbb{HP}^k$. Moreover, we can take f to be transverse to $\mathbb{HP}^{k-1} \subset \mathbb{HP}^k$. Set $\sigma(M) := f^{-1}(\mathbb{HP}^{k-1})$.

The candidate manifold $\sigma(M)$ is constructed in the same manner as Bahri–Gilkey's Smith isomorphism for Spin^c and Pin^{c-} cobordism [BG87, Lemma 3.1(a)]. In the Spin^c setting, checking that $\sigma(M)$ is a Pin^{c-} manifold is a single characteristic class computation. We do not have an analogous result for determining the existence of Pin^{h-} structure, so new ideas are needed to continue this approach.

10.4. **Conner–Floyd surjection.** One important application of the Anderson–Brown–Peterson splitting of MSpin and MSpin^c is in the work of Hopkins and Hovey [HH92], who proved that $\mathrm{MSpin}_*(-)$ and $\mathrm{MSpin}^c_*(-)$ satisfy Conner–Floyd isomorphisms with respect to $\mathrm{KO}_*(-)$ and $\mathrm{KU}_*(-)$.

Theorem 10.10 (Hopkins–Hovey)**.** *The Atiyah–Bott–Shapiro orientations* $\varphi^r : \mathrm{MSpin} \to \mathrm{KO}$ *and* $\varphi^c : \mathrm{MSpin}^c \to \mathrm{KU}$ *induce maps*

$$\mathrm{MSpin}_*(X) \otimes_{\mathrm{MSpin}_*} \mathrm{KO}_* \to \mathrm{KO}_*(X),$$
$$\mathrm{MSpin}^c_*(X) \otimes_{\mathrm{MSpin}^c_*} \mathrm{KU}_* \to \mathrm{KU}_*(X)$$

that are natural isomorphisms of KO_**- and* KU_**-modules, respectively, for all spectra* X*.*

It is natural to wonder whether an analog holds for $\mathrm{MSpin}^h_*(-)$ with respect to $\mathrm{KSp}_*(-)$. One obvious wrinkle is that MSpin^h_* is not itself a ring, but rather a module over MSpin_*. It turns out that we get a Conner–Floyd surjection, but not an isomorphism [Hu23, Theorem 6.1.1]:

Theorem 10.11 (Hu)**.** *The Atiyah–Bott–Shapiro map* $\varphi^h : \mathrm{MSpin}^h \to \mathrm{KSp}$ *induces a surjection*

$$\mathrm{MSpin}^h_*(X) \otimes_{\mathrm{MSpin}_*} \mathrm{KO}_* \to \mathrm{KSp}_*(X)$$

for all spectra X*. Moreover, this surjection admits a canonical splitting that is natural in* X*.*

Since the splitting of $\mathrm{MSpin}_*(X) \otimes_{\mathrm{MSpin}_*} \mathrm{KO}_* \to \mathrm{KSp}_*(X)$ is natural in X, one might hope to characterize the kernel in terms of X.

Problem 10.12. Characterize the kernel of $\mathrm{MSpin}^h_*(X) \otimes_{\mathrm{MSpin}_*} \mathrm{KO}_* \to \mathrm{KSp}_*(X)$.

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Table 2. $\pi_n\mathrm{MSpin} \cong \mathbb{Z}^r \times (\mathbb{Z}/2\mathbb{Z})^t$.

n	r	t	n	r	t	n	r	t	n	r	t	n	r	t
0	1	0	20	7	1	40	42	4	60	176	67	80	627	343
1	0	1	21	0	0	41	0	45	61	0	38	81	0	931
2	0	1	22	0	1	42	0	60	62	0	80	82	0	1196
3	0	0	23	0	0	43	0	2	63	0	36	83	0	330
4	1	0	24	11	0	44	56	14	64	231	70	84	792	589
5	0	0	25	0	11	45	0	6	65	0	290	85	0	448
6	0	0	26	0	15	46	0	17	66	0	379	86	0	698
7	0	0	27	0	0	47	0	4	67	0	58	87	0	494
8	2	0	28	15	2	48	77	11	68	297	142	88	1002	721
9	0	2	29	0	1	49	0	86	69	0	90	89	0	1658
10	0	3	30	0	3	50	0	114	70	0	169	90	0	2103
11	0	0	31	0	0	51	0	7	71	0	92	91	0	729
12	3	0	32	22	1	52	101	31	72	385	158	92	1255	1171
13	0	0	33	0	23	53	0	15	73	0	521	93	0	952
14	0	0	34	0	31	54	0	38	74	0	676	94	0	1385
15	0	0	35	0	0	55	0	13	75	0	143	95	0	1068
16	5	0	36	30	6	56	135	29	76	490	291	96	1575	1472
17	0	5	37	0	2	57	0	159	77	0	205	97	0	2948
18	0	7	38	0	7	58	0	210	78	0	347	98	0	3689
19	0	0	39	0	1	59	0	22	79	0	219	99	0	1550

TABLE 3. $\pi_n \mathrm{MSpin}^c \cong \mathbb{Z}^r \times (\mathbb{Z}/2\mathbb{Z})^t$.

n	r	t	n	r	t	n	r	t	n	r	t	n	r	t
0	1	0	20	19	1	40	139	26	60	684	284	80	2714	2152
1	0	0	21	0	0	41	0	8	61	0	148	81	0	1490
2	1	0	22	19	5	42	139	59	62	684	458	82	2714	2986
3	0	0	23	0	0	43	0	10	63	0	184	83	0	1820
4	2	0	24	30	2	44	195	44	64	915	434	84	3506	3145
5	0	0	25	0	0	45	0	16	65	0	243	85	0	2268
6	2	0	26	30	9	46	195	90	66	915	676	86	3506	4273
7	0	0	27	0	0	47	0	20	67	0	301	87	0	2762
8	4	0	28	45	4	48	272	72	68	1212	658	88	4508	4564
9	0	0	29	0	1	49	0	29	69	0	391	89	0	3418
10	4	1	30	45	14	50	272	138	70	1212	987	90	4508	6095
11	0	0	31	0	1	51	0	36	71	0	483	91	0	4147
12	7	0	32	67	8	52	373	116	72	1597	985	92	5763	6583
13	0	0	33	0	2	53	0	51	73	0	619	93	0	5099
14	7	1	34	67	24	54	373	207	74	1597	1436	94	5763	8651
15	0	0	35	0	2	55	0	64	75	0	762	95	0	6167
16	12	0	36	97	15	56	508	183	76	2087	1462	96	7338	9440
17	0	0	37	0	4	57	0	88	77	0	967	97	0	7540
18	12	3	38	97	37	58	508	311	78	2087	2074	98	7338	12237
19	0	0	39	0	5	59	0	110	79	0	1186	99	0	9090

Table 4. $\pi_n \mathrm{MSpin}^h \cong \mathbb{Z}^r \times (\mathbb{Z}/2\mathbb{Z})^t$.

n	r	t
0	1	0
1	0	0
2	0	0
3	0	0
4	2	0
5	0	2
6	0	2
7	0	0
8	4	0
9	0	1
10	0	1
11	0	0
12	7	0
13	0	7
14	0	8
15	0	2
16	12	1
17	0	3
18	0	3
19	0	1

n	r	t
20	19	2
21	0	21
22	0	25
23	0	7
24	30	5
25	0	10
26	0	11
27	0	7
28	45	10
29	0	55
30	0	64
31	0	22
32	67	20
33	0	31
34	0	35
35	0	27
36	97	36
37	0	132
38	0	156
39	0	66

n	r	t
40	139	65
41	0	87
42	0	100
43	0	86
44	195	111
45	0	307
46	0	360
47	0	180
48	272	188
49	0	232
50	0	269
51	0	249
52	373	310
53	0	689
54	0	804
55	0	465
56	508	503
57	0	592
58	0	685
59	0	662

n	r	t
60	684	803
61	0	1514
62	0	1755
63	0	1154
64	915	1267
65	0	1445
66	0	1663
67	0	1659
68	1212	1972
69	0	3273
70	0	3767
71	0	2746
72	1597	3039
73	0	3402
74	0	3891
75	0	3968
76	2087	4636
77	0	6971
78	0	7962
79	0	6315

n	r	t
80	2714	7010
81	0	7757
82	0	8808
83	0	9121
84	3506	10510
85	0	14645
86	0	16609
87	0	14094
88	4508	15640
89	0	17174
90	0	19367
91	0	20280
92	5763	23104
93	0	30368
94	0	34201
95	0	30607
96	7338	33906
97	0	37043
98	0	41508
99	0	43818