Local Contributions in $\mathbb{A}^1$-Enumerative Geometry

by

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Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Mathematics
in the Graduate School of Duke University
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Abstract

Bézout's theorem is a fundamental result in enumerative geometry: over an algebraically closed field, the intersection of n general hypersurfaces in $\mathbb{P}^n$ of degrees $d_1, \ldots, d_n$ consists of $d_1 \cdots d_n$ points, provided that one counts these intersection points with the appropriate multiplicity.

Bézout's theorem implies other classical theorems from enumerative geometry, such as the count of the circles of Apollonius. Given three circles in general position, there are eight circles that are simultaneously tangent to the original three. From a Bézout-theoretic perspective, this is because the space of circles tangent to a given circle is a quadratic cone in $\mathbb{P}^3$, and the intersection of three quadratic cones in $\mathbb{P}^3$ consists of $2^3 = 8$ points.

We prove versions of Bézout's theorem and the circles of Apollonius over non-algebraically closed fields. Our work follows the general theme of the $\mathbb{A}^1$-enumerative geometry program as initiated by Kass–Wickelgren, Levine, and others. We express a global *fixed count* as a sum of *local contributions*, where the local contributions depend on the objects being enumerated. Working over a field k, both the fixed count and local contributions are valued in the Grothendieck–Witt ring $\mathrm{GW}(k)$ of isomorphism classes of non-degenerate symmetric bilinear forms over k. By taking relevant invariants of bilinear forms over k, we obtain a weighted count of intersection points over k.

While there have been significant developments in the literature on computing

fixed counts in $\mathbb{A}^1$-enumerative geometry, the story of local contributions is non-plussing. Our work on Bézout's theorem and the circles of Apollonius focuses especially on describing the local contributions geometrically. We pose the *geometricity problem*, which asks whether one can construct a geometric taxonomy for local contributions in $\mathbb{A}^1$-enumerative geometry. We show how Bézout's theorem answers a naïve version of the geometricity problem, and we use the circles of Apollonius to explain why Bézout's theorem does not answer a more interesting version of geometricity.

For Bézout's theorem, we use the $\mathbb{A}^1$-degree to associate a bilinear form (up to isomorphism) to each intersection point, with the rank of the quadratic form given by the intersection multiplicity. At transverse intersection points, this bilinear form is determined by the gradient vectors of the hypersurfaces. At non-transverse intersection points, one can use a deformation to express the bilinear form as a direct sum over transverse intersections. Using an Euler class from motivic homotopy theory, we show that the direct sum of these "intersection forms" is hyperbolic of rank $d_1 \cdots d_n$.

Our $\mathbb{A}^1$-enumerative version of Bézout's theorem gives the global fixed count for the circles of Apollonius. We also show that the Bézout-theoretic local contribution can be viewed as a universal, albeit unsatisfactory, local contribution in $\mathbb{A}^1$-enumerative geometry. By giving a geometric description of the local contributions associated to the circles of Apollonius, we illustrate the shortcomings of Bézout's theorem as a universal local contribution.

Dedication

To Eden and Tessa.

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Acknowledgements

Throughout my academic journey, I have been incredibly blessed by the many kind and generous people who have given me their support. First and foremost, I thank my wife Eden for her love and dedication. Her selfless character is an inspiration to me, and I consider myself lucky beyond measure to have her in my life. I thank my sweet daughter Tessa for the many wonderful distractions, and for being the joy of my life. My cat Finn sat on me through most of the writing portion of this dissertation, and I am grateful for his company.

I thank my parents for the many sacrifices they have made to raise their children. Their example has led me throughout my life. Thanks to my siblings Billy and Christie, Jon and Tricia, James and Aleisha, Anna and Josh, Michael and Misha, and Kathryn for their encouragement, and for taking interest in my research. I am also grateful to Matt and Tana, Alex and Charlotte, Brooke, Preston, and Rowan for welcoming me into their family and for their support.

Next, I owe my deepest gratitude to Kirsten Wickelgren. Her guidance, insights, and encouragement have made me a better mathematician and educator. I am also grateful for the example she sets in tirelessly serving both her family and the academic community. I can only hope to emulate her in this regard.

I thank Don Tucker for sparking my interest in mathematics, as well as Henryk Hecht, Sri Iyengar, Stefan Patrikis, and Anurag Singh for their excellent mentorship during my undergraduate years and beyond. I would not have been able to become a mathematician without their influence. Thanks also to Mladen Bestvina for encouraging me to ask more questions, Evelyn Lamb for telling me to get past point-set before giving up on topology, Joseph Metz for suggesting I pursue a graduate degree (even if it did not end up being in Germanics), and Dragan Miličić for teaching me to see the forest through the trees.

Thanks to John Etnyre, Jen Hom, and Dan Margalit for welcoming me into the topology group at Georgia Tech, Joe Rabinoff for teaching me everything I know about schemes, and Greg Blekherman for giving me a taste for computational algebraic geometry.

I moved to Duke just before the pandemic broke out in the US. I am grateful to Paul Aspinwall, Clark Bray, Robert Bryant, Veronica Ciocanel, Samit Dasgupta, Kristen Gerondelis, Dick Hain, Adam Levine, Jianfeng Lu, Lenny Ng, Colleen Robles, Les Saper, and Shira Viel for making the department a welcoming and supportive place during such a strange time. Thanks also to the department staffs at Utah, Georgia Tech, and Duke for their excellent work that too often goes unnoticed.

To my coauthors Thomas Brazelton, Robert Burklund, Cameron Darwin, Aygul Galimova, Pam Gu, Daniel Minahan, Michael Montoro, Morgan Opie, Sabrina Pauli, Soumya Sankar, and Tianyi Zhang, I am grateful for the many exciting and enriching discussions, mathematical and otherwise. My particular thanks to Thomas Brazelton and Sabrina Pauli for their extended collaboration — I learn something new every time I talk with them.

I am indebted to the broader mathematical community for its magnanimity. Beyond the aforementioned mathematicians, I have had the good fortune of learning from Tom Bachmann, Clark Barwick, Federico Binda, Angie Cueto, Frédéric Déglise, Igor Dolgachev, Benson Farb, Marc Hoyois, Dan Isaksen, Jesse Kass, Viatcheslav Kharlamov, Antonio Lerario, Max Lieblich, Daniel Litt, Dan Loughran, Kyle Ormsby, Paul Arne Østvær, Jean-Pierre Serre, Matthias Wendt, Jesse Wolfson, Ravi Vakil, and many others.

Finally, I am grateful to God, whose constant and abundant love and guidance have been apparent in the wonderful people that have been placed in my life.

This work was partially supported by Kirsten Wickelgren's NSF CAREER grant (DMS-1552730).

1

Introduction

There is a single line through any pair of points, eight circles simultaneously tangent to a given trio of circles, and twenty-seven lines on any smooth cubic surface. For millennia, mathematicians have sought to count solutions to geometric questions. Traditionally, most results in enumerative geometry require that one work over an algebraically closed field. Over non-algebraically closed fields, *invariance of count* is lost: there are generally multiple possible counts for a given enumerative question. While interesting things can be said about these possible counts [23,47,59], one would also like a fixed count over any given field.

The path forward is suggested by real enumerative geometry. By the celebrated theorem of Cayley and Salmon [15], every smooth cubic surface over $\mathbb{C}$ contains 27 lines. In contrast, Schläfli showed that a smooth cubic surface over $\mathbb{R}$ may contain 3, 7, 15, or 27 lines [58]. Segre noted that there are two types of lines on real cubic surfaces (hyperbolic and elliptic), but it was not until much later that Finashin–Kharlamov and Okonek–Teleman noted that there are always exactly 3 more hyperbolic lines than elliptic lines on any real smooth cubic surface [30, 54]. That is, $\#\{\text{hyperbolic lines}\} - \#\{\text{elliptic lines}\} = 3$. In particular, when real lines on real cubic surfaces are counted with an appropriate weight, invariance of count is restored.

The goal of the $\mathbb{A}^1$-*enumerative geometry* program, also known as *enriched* or *quadratic* enumerative geometry, is to restore invariance of count for enumerative geometric problems over non-algebraically closed fields (or even more general base schemes). Instead of integer-valued counts, $\mathbb{A}^1$-enumerative geometry provides counts valued in isomorphism classes of symmetric non-degenerate bilinear forms. While a bilinear form-valued count may initially seem bizarre or unnecessarily abstract, this approach offers a distinct advantage: rather than working over a specific base field, these quadratic enumerative formulas are valid over an arbitrary base field (perhaps subject to some technical hypotheses). One can then recover a fixed, weighted count over a given field k by applying relevant invariants of bilinear forms over k.

This paradigm is illustrated by Kass and Wickelgren's enriched count of lines on smooth cubic surfaces [37]. Let $\mathrm{GW}(k)$ be the Grothendieck–Witt ring of isomorphism classes of non-degenerate symmetric bilinear forms over a field k. This ring is generated by the rank 1 forms $\langle a\rangle = [(x, y) \mapsto axy]$, where $a \in k^\times$; addition and multiplication are given by direct sum and tensor product, respectively. To each line L on a smooth cubic surface X over k, Kass and Wickelgren associate a non-zero scalar a_L in the field of definition $k(L)$ of the line. They then show that

$$\sum_{L\subset X} \mathrm{Tr}_{k(L)/k}\langle a_L\rangle = 15\langle 1\rangle + 12\langle -1\rangle, \tag{1.1}$$

where $\mathrm{Tr}_{k(L)/k}$ denotes the field trace. Over an algebraically closed field, the rank of Equation 1.1 states that $\#\{L \subset X\} = 15 + 12$, recovering the classical result. Over $\mathbb{R}$, taking the signature of Equation 1.1 yields

$$\#\{L : \mathbb{R}(L) = \mathbb{R} \text{ and } a_L > 0\} - \#\{L : \mathbb{R}(L) = \mathbb{R} \text{ and } a_L < 0\} = 15 - 12.$$

By showing that $a_L > 0$ for hyperbolic lines and $a_L < 0$ for elliptic lines, Kass and

Wickelgren recover the fixed, weighted count of real lines on smooth cubic surfaces. However, one need not stop at the already-known cases of $\mathbb{C}$ and $\mathbb{R}$. Over any finite field of odd characteristic, Kass and Wickelgren take the discriminant of Equation 1.1 to deduce that the number of elliptic lines defined over an odd extension plus the number of hyperbolic lines defined over an even extension is always an even number.

In order to make sense of the latter result, one needs to define what it means for a line to be elliptic or hyperbolic over a finite field. In analogy with the real case, one could define a line L to be *hyperbolic* if a_L is a square in $k(L)^\times$ and *elliptic* otherwise. However, this algebraic definition lacks the geometric nature of hyperbolic and elliptic lines on real cubic surfaces. One of the key results in Kass and Wickelgren's enriched count of lines on cubic surfaces is their characterization of the local index $\langle a_L \rangle$ in terms of the geometry of the line L.

Geometrically interpreting the local indices that arise in $\mathbb{A}^1$-enumerative geometry will be a central theme in this dissertation. While there are many readily available tools for computing local indices (see Section 2.1), finding a geometric description of the local index is not so straightforward. This leads us to the following questions: are local indices in $\mathbb{A}^1$-enumerative geometry always geometric, and, if so, is there a classification of enumerative problems in terms of their local geometric interpretation? We call these two questions the *geometricity* problem (see Question 5.1), which we discuss in more detail in Chapter 5.

We should say something about the prefix "$\mathbb{A}^1$," which comes from the technical machinery underlying $\mathbb{A}^1$-enumerative geometry. In practice, the local index is given by the isomorphism class of the Scheja–Storch bilinear form [64]. By the work of Eisenbud–Levine [27] and, independently, Khimshiashvili [34], the Scheja–Storch form can be computed in terms of a special socle element of a local Artin ring determined by the given morphism. More recently, we showed in joint work with Brazelton and Pauli that the Scheja–Storch form can be computed using multivariate Bézoutians [12]. Both of these approaches can be phrased purely in terms of commutative algebra, and the latter approach has been implemented in Sage [11].

Commutative algebraic formulas for local indices are convenient, but the true power of the theory comes from its topological guise. In a landmark result, Kass and Wickelgren showed that the bilinear form of Eisenbud, Levine, and Khimshiashvili is in fact the local $\mathbb{A}^1$-degree from $\mathbb{A}^1$-homotopy theory [36]. This homotopy theory (also known as *motivic* homotopy theory) enables one to import many tools from classical topology to the world of algebraic geometry [53]. In particular, motivic homotopy theory provides a $\mathrm{GW}(k)$-valued Euler class for vector bundles on schemes. Using an analog of the Poincaré–Hopf theorem, Kass and Wickelgren relate the local contributions coming from a sum of local $\mathbb{A}^1$-degrees to a global fixed count in the form of an Euler class [37]. In other words, it is this connection to topology that allows us to restore invariance of count. We will discuss these foundations in more detail in Chapter 2.

There are many methods for computing Euler classes (and other characteristic classes) in classical topology. Finding motivic homotopical analogs of these methods offers new tools for computing fixed counts in $\mathbb{A}^1$-enumerative geometry, and this is an active area of research [2, 3, 41–43, 55]. In this sense, fixed counts in $\mathbb{A}^1$-enumerative geometry come with an inherent interpretation but are hard to compute. Our narrative for this dissertation will focus instead on local contributions, which are relatively easy to compute but are hard to interpret.

1.1 Overview of results

In this dissertation, we give an $\mathbb{A}^1$-enumerative treatment of two classical results and discuss their relationship to the geometricity problem. The first is Bézout's theorem, which counts intersection points of projective hypersurfaces.

Theorem 1.1 (Bézout's theorem)**.** *Let $X_1, \ldots, X_n \subset \mathbb{P}^n$ be general hypersurfaces of degrees $d_1, \ldots, d_n$ over an algebraically closed field. Then $\bigcap_{i=1}^n X_i$ consists of $d_1 \cdots d_n$ points (when counted with intersection multiplicity).*

Over a non-algebraically closed field, Theorem 1.1 is no longer true as stated. For example, the curves in $\mathbb{P}^2$ defined by $yz^2 - x^3$ an $x^2 + y^2 - z^2$ have only two multiplicity one intersections over $\mathbb{R}$ (see Figure 1.1). This can be rectified by including non-rational intersections and weighting them by the degree of their residue field over the base, but this is in essence just a restatement of Theorem 1.1.

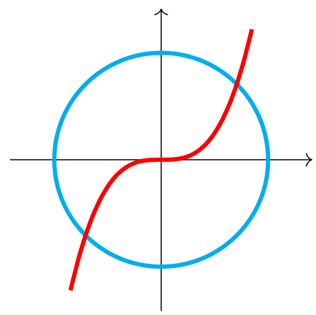

FIGURE 1.1: Two curves over $\mathbb{R}$

We give an enrichment of Bézout's theorem. We assume $\sum_{i=1}^n d_i \equiv n+1 \mod 2$ for technical reasons, and we begin with an assumption that $X_1, \ldots, X_n$ intersect transversely. This allows us to geometrically interpret the relevant local index as an "intersection volume." We then use a dynamic version of the $\mathbb{A}^1$-degree to remove the transversality assumption in Chapter 5.

Theorem 1.2 (See Theorem 3.2)**.** *Let $\sum_{i=1}^n d_i \equiv n+1 \mod 2$, and let $X_1, \ldots, X_n$ be hypersurfaces of $\mathbb{P}^n$ of degree $d_1, \ldots, d_n$ that intersect transversely. Given an intersection point p of $X_1, \ldots, X_n$, let $J(p)$ be the signed volume of the parallelpiped determined by the gradient vectors of $X_1, \ldots, X_n$ at p. Then summing over the intersection points of $X_1, \ldots, X_n$, we have*

$$\sum_{points} \mathrm{Tr}_{k(p)/k} \langle J(p) \rangle = \frac{d_1 \cdots d_n}{2} (\langle 1 \rangle + \langle -1 \rangle),$$

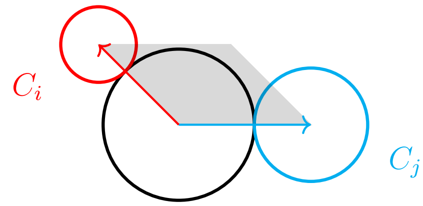

FIGURE 1.2: Geometric interpretation for circles of Apollonius

where $\mathrm{Tr}_{k(p)/k} : \mathrm{GW}(k(p)) \to \mathrm{GW}(k)$ *is given by post-composing with the field trace.*

Applying invariants such as rank, signature, and discriminant allows us to deduce analogs of Bézout's theorem over fields like $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{F}_q$.

Classically, the circles of Apollonius are a special case of Bézout's theorem. The space of circles tangent to a given circle is a quadric cone in $\mathbb{P}^3$, so the circles of Apollonius correspond to the 2^3 intersection points of three such cones. This allows us to immediately deduce the beginnings of an enriched count of the circles of Apollonius from Theorem 1.2. However, the geometric interpretation coming from Bézout's theorem is phrased in terms of the parameter cones instead of in terms of the circles themselves. The main result of Chapter 4 gives a more geometrically intrinsic interpretation of the local indices.

Lemma 1.3 (See Lemma 4.17)**.** *The local index for a circle C tangent to C_1, C_2, C_3 is given by $\mathrm{Tr}_{k(C)/k}\langle P(C)\rangle$, where $P(C)$ is an alternating sum of the areas of the parallelograms spanned by the center of C and the centers of C_i and C_j for $1 \leqslant i < j \leqslant 3$ (see Figure 1.2).*

In analogy with the dynamic local $\mathbb{A}^1$-degree of Pauli–Wickelgren (which we recall in Theorem 2.26), we give a description of the local $\mathbb{A}^1$-degree in families.

Theorem 1.4 (See Theorem 2.28)**.** *Let $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ with isolated zero p such that $k(p)/k$ separable. Let $F : \mathbb{A}^n_{k[t]} \to \mathbb{A}^n_{k[t]}$ be a morphism such that $\mathbb{V}(F) \to \operatorname{Spec} k[t]$ is flat and $F|_{t=0} = f$. Assume that $\mathbb{V}(F)$ is unramified away from $t = 0$. Then for any* *closed point* $c \in \mathbb{A}^1_k$*, the perturbation* $\tilde{f} := F|_{t=c} : \mathbb{A}^n_k \to \mathbb{A}^n_k$ *of* f *has a set of zeros* $Z \subseteq \tilde{f}^{-1}(0)$ *such that*

$$\deg_p^{\mathbb{A}^1}(f) = \sum_{q \in Z} \deg_q^{\mathbb{A}^1}(\tilde{f}).$$

In forthcoming work, we will use Theorem 1.4 to give further geometric interpretations of the local indices for the circles of Apollonius.

In Chapter 5, we ask whether all enriched enumerative problems admit a geometric interpretation of their local indices (see Question 5.1). We show that Bézout's theorem gives a universal geometric description in enriched enumerative geometry.

Proposition 1.5 (See Proposition 5.5)**.** *Let* X *be a* k*-scheme of dimension* n*. Let* $V \to X$ *be a relatively orientable vector bundle of rank* n*, and let* $\rho : \det V \otimes \omega_X \xrightarrow{\cong} L^{\otimes 2}$ *be a relative orientation of* $V \to X$*. Let* $\sigma : X \to V$ *be a section. If* $p \in \sigma^{-1}(0)$ *is a simple zero with separable residue field* $k(p)/k$*, then the local index* $\mathrm{ind}_p\,\sigma$ *is equal to the intersection volume* $\mathrm{Tr}_{k(p)/k}\langle \mathrm{Vol}(p)\rangle$*.*

Using the dynamic local degree (Corollary 5.6) and the familial local degree (Corollary 5.7), we remove the transversality assumption from Proposition 1.5. However, as a case study, the circles of Apollonius indicate that the universal geometric interpretation coming from Bézout's theorem is not a satisfactory answer to Question 5.1. Indeed, we are interested in interpretations that are phrased in terms of the relevant geometry, rather than in terms of the moduli spaces parameterizing the objects being counted. This demonstrates how the local information in $\mathbb{A}^1$-enumerative geometry is strictly richer than that of classical enumerative geometry.

1.2 Notation

The following notation will be used throughout this dissertation. Notation specific to a given chapter will be given in the relevant chapter. We denote by k an arbitrary field. The ring of power series and the field of Laurent series over k will be denoted by $k[\![t]\!]$ and $k(\!(t)\!)$, respectively. The vanishing locus of a collection of polynomials (respectively, homogeneous polynomials) $f_1, \ldots, f_m$, considered as a subvariety of affine space (respectively, projective space) is denoted $\mathbb{V}(f_1, \ldots, f_m)$.

1.2.1 Grothendieck–Witt groups

The *Grothendieck–Witt group* $\mathrm{GW}(k)$ is the group completion of the monoid of isomorphism classes of symmetric, non-degenerate bilinear forms over k, where the group law is given by direct sum. The Grothendieck-Witt group is in fact a ring, where multiplication comes from the tensor product of bilinear forms. See e.g. [39] for the case where $\operatorname{char} k \neq 2$.

Given $a \in k^\times$, we denote by $\langle a\rangle$ the isomorphism class of the bilinear form $(x, y) \mapsto axy$. It is a fact that $\mathrm{GW}(k)$ is generated by all such $\langle a\rangle$, subject to the following relations [39, Chapter II, Theorem 4.1]:

(i) $\langle ab^2\rangle = \langle a\rangle$ for all $a, b \in k^\times$.

(ii) $\langle a\rangle\langle b\rangle = \langle ab\rangle$ for all $a, b \in k^\times$.

(iii) $\langle a\rangle + \langle b\rangle = \langle a + b\rangle + \langle ab(a + b)\rangle$ for all $a, b \in k^\times$ such that $a + b \neq 0$.

(iv) $\langle a\rangle + \langle -a\rangle = \langle 1\rangle + \langle -1\rangle$ for all $a \in k^\times$.

Relation (iv) is actually redundant, but it is useful to know. We will use the notation $\mathbb{H} := \langle 1\rangle + \langle -1\rangle$, as this bilinear form (called the *hyperbolic form*) will appear frequently. By relations (ii) and (iv), we note that $\langle a\rangle\cdot\mathbb{H} = \mathbb{H}$ and $\mathbb{H}\cdot\mathbb{H} = 2\cdot\mathbb{H}$. Both the ring multiplication of $\mathrm{GW}(k)$ and the integer multiplication of $\mathrm{GW}(k)$ as an abelian group under addition may be denoted by $\cdot$ or by juxtaposition of symbols, whichever is presently more visually appealing or less confusing.

Given a finite separable field extension L/k, the field trace $\mathrm{Tr}_{L/k} : L \to k$ is defined by $\mathrm{Tr}_{L/k}(x) = \sum_{\sigma \in \mathrm{Gal}(L/k)} \sigma(x)$. Given a bilinear form $V \times V \to L$, one can post-compose with $\mathrm{Tr}_{L/k}$ to obtain a bilinear form $V \times V \to L \to k$. It follows that post-composition with the field trace induces a transfer $\mathrm{Tr}_{L/k} : \mathrm{GW}(L) \to \mathrm{GW}(k)$.

1.2.2 $\mathbb{A}^1$-degrees

The Brouwer degree in algebraic topology is often first defined as the group isomorphism $[S^n, S^n] \cong \mathbb{Z}$. There is an analogous construction in motivic homotopy theory. There are two motivic circles, namely the simplicial circle S^1 and the multiplicative group sheaf $\mathbb{G}_m$, so spheres in this context are of the form $S^a \wedge \mathbb{G}_m^{\wedge b}$. In one particular family of spheres, there is a convenient model in terms of schemes: $S^n \wedge \mathbb{G}_m^{\wedge n} \simeq \mathbb{P}^n_k/\mathbb{P}^{n-1}_k$. Morel gives a group (in fact, ring) homomorphism

$$[\mathbb{P}^n_k/\mathbb{P}^{n-1}_k, \mathbb{P}^n_k/\mathbb{P}^{n-1}_k] \to \mathrm{GW}(k) \tag{1.2}$$

that is a surjection for $n = 1$ and an isomorphism for $n \geqslant 2$ [52]. Morel's homomorphism is called the $\mathbb{A}^1$-*degree*, which we will denote by $\deg^{\mathbb{A}^1}$ or simply deg if no confusion arises. One can also associate a *local* $\mathbb{A}^1$-*degree* to a map $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ at a zero p (see 2.1.1 for details). We denote the local $\mathbb{A}^1$-degree of f at p by $\deg^{\mathbb{A}^1}_p(f)$ or $\deg_p(f)$.

1.3 Outline

We give a survey of theoretical and computational tools for $\mathbb{A}^1$-enumerative geometry in Chapter 2, as well as a description of the dynamic $\mathbb{A}^1$-degree of Pauli and Pauli–Wickelgren [55, 56]. We discuss the local $\mathbb{A}^1$-degree in families in Section 2.1.3. In Chapter 3, we give an arithmetic enrichment of Bézout's theorem (which previously appeared in [45]). We then discuss the circles of Apollonius in the $\mathbb{A}^1$-enumerative context in Chapter 4. In Chapter 5, we pose the *geometricity* problem for local indices in $\mathbb{A}^1$-enumerative geometry, and we discuss the insights about this problem gained from Chapters 3 and 4. Finally, we review the results of this dissertation in Chapter 6.

2

$\mathbb{A}^1$-Enumerative Geometry

We now give a brief outline of some of the tools and techniques available within $\mathbb{A}^1$-enumerative geometry (see also [8, 37, 56]). Our focus will be on the bundle-theoretic approach to enumerative geometry: we take a moduli space X parameterizing the types of objects we want to count (over a field k), as well as a vector bundle $V \to X$ that encodes the geometric conditions we wish to impose on our objects of study. We then construct a section $\sigma : X \to V$ that vanishes precisely on the objects that satisfy the desired geometric conditions.

Example 2.1. We can frame Bézout's theorem as follows. Sections of $\mathcal{O}(d_1) \oplus \cdots \oplus \mathcal{O}(d_n) \to \mathbb{P}^n$ are given by n-tuples $(f_1, \ldots, f_n)$ of homogeneous polynomials of degrees $d_1, \ldots, d_n$ in $n+1$ variables. Such a section vanishes on a point $p \in \mathbb{P}^n$ if and only if p is contained in the intersection $\mathbb{V}(f_1) \cap \cdots \cap \mathbb{V}(f_n)$. Note that the particular choice of hypersurfaces $\mathbb{V}(f_1), \ldots, \mathbb{V}(f_n)$ depends on the choice of section σ.

Using an analog of the Poincaré–Hopf theorem, we can express the Euler class $e(V, \sigma) \in \mathrm{GW}(k)$ as a sum of local indices:

$$e(V,\sigma) = \sum_{p\in\sigma^{-1}(0)} \operatorname{ind}_p \sigma. \tag{2.1}$$

In order to prove an enumerative theorem, we must address both sides of Equation 2.1. As described in Example 2.1, the choice of section $\sigma : X \to V$ corresponds to a particular instance of the enumerative problem; by showing that $e(V,\sigma)$ does not depend on σ, we obtain a global fixed count. Similarly, we need to show that $\operatorname{ind}_p \sigma$ (however it is defined) does not depend on any choices made in the general setup. Finally, we wish to geometrically describe the local indices $\operatorname{ind}_p \sigma$ in terms of the relevant problem. See Sections 3.4 and 4.4 for examples and Chapter 5 for a general discussion of such geometric interpretations.

Throughout our discussion, we will assume that X is a smooth k-scheme of dimension at least 1. The dimension assumption ensures that local coordinates of the desired form exist. The smoothness assumption can be relaxed if one works over a smooth subscheme of X. One may work over base schemes more general than just fields in motivic homotopy theory, and much of the relevant setup for $\mathbb{A}^1$-enumerative geometry in this context is given by Bachmann–Wickelgren [2]. There is also recent work of Khan–Ravi and Chowdhury on motivic homotopy theory for algebraic stacks [19,38], so the requirement that X be a scheme should be removable as well.

2.1 Local contributions

We begin by recalling the definition of the local index $\operatorname{ind}_p \sigma$ of a section $\sigma : X \to V$, with [37, Section 4] as the standing reference for this section. The general idea is to use local coordinates on X and a local trivialization of V to rewrite σ as a map $\mathbb{A}^n_k \to \mathbb{A}^n_k$, where $n = \dim X = \operatorname{rank} V$. The local index $\operatorname{ind}_p \sigma$ will then be defined as the local $\mathbb{A}^1$-degree of this map at the image of p under the local coordinates. After going through this setup, we will recall the definition of and tools for computing the local $\mathbb{A}^1$-degree.

Definition 2.2. *Nisnevich coordinates* about a closed point $p \in X$ consist of a Zariski open neighborhood $U \subseteq X$ containing p, along with an étale map $\varphi : U \to \mathbb{A}^n_k$ that induces an isomorphism $k(p) \cong k(\varphi(p))$ of residue fields.

Using the smoothness assumption on X, Kass and Wickelgren show that Nisnevich coordinates exist about any closed point $p \in X$ when $\dim X \geqslant 1$ [37, Proposition 20]. For our next step, we would like a local inverse $\varphi^{-1} : \mathbb{A}^n_k \to U$ so that we can write $\sigma \circ \varphi^{-1} : \mathbb{A}^n_k \to V|_U$. Often, one can pick an isomorphism $\varphi : U \to \mathbb{A}^n_k$, from which one automatically gets Nisnevich coordinates about p and the desired local inverse. In general, φ is étale-locally a finite cover. By assuming that p is an *isolated* zero (in the sense of the following definition), we can therefore define a local inverse to φ on U.

Definition 2.3. A closed point $p \in X$ is called an *isolated zero* of a section $\sigma : X \to V$ if there exists a Zariski open neighborhood U of p such that, as sets, we have $U \cap \mathbb{V}(\sigma) = \{p\}$. Equivalently, the local ring $\mathcal{O}_{\mathbb{V}(\sigma),p}$ is a finite k-algebra [37, Definition 22 and Proposition 23]. We say that σ has *isolated zeros* if any of the following three equivalent criteria hold:

(i) all zeros of σ are isolated,

(ii) $\mathbb{V}(\sigma)$ consists of finitely many closed points, or

(iii) $\mathcal{O}_{\mathbb{V}(\sigma)}$ is a finite k-algebra.

Now that we have $\sigma \circ \varphi^{-1} : \mathbb{A}^n_k \to V|_U$, we will use a local trivialization $V|_U \to U \times \mathbb{A}^n_k \xrightarrow{\mathrm{proj}_2} \mathbb{A}^n_k$ to build our morphism $\mathbb{A}^n_k \to \mathbb{A}^n_k$. (If necessary, we can intersect

U an appropriately small neighborhood of p to ensure that $V|_U$ is trivial.) However, not every possible choice of trivialization $\psi : V|_U \to \mathbb{A}^n_k$ is suitable for our purposes. We want the local $\mathbb{A}^1$-degree $\deg_{\varphi(p)}(\psi \circ \sigma \circ \varphi^{-1})$ to be independent of the choice of coordinates (U, φ) and trivialization ψ. Because squares are trivial in $\mathrm{GW}(k)$, the following definitions give us the desired conditions on our coordinates and trivialization [37, Definitions 17 and 21].

Definition 2.4. A *relative orientation* of $V \to X$ is a pair (L, rho), where L is a line bundle and $rho : L^{\otimes 2} \to \mathrm{Hom}(\det TX, \det V)$. If $V \to X$ has a relative orientation, then we say that V is *relatively orientable*.

Definition 2.5. A section $s \in \Gamma(U, \mathrm{Hom}(\det TX, \det V))$ is called a *square* if its preimage $rho^{-1}(s) \in \Gamma(U, L^{\otimes 2})$ is of the form $\ell \otimes \ell$ for some section $\ell \in \Gamma(U, L)$.

Definition 2.6. Nisnevich coordinates $\varphi = (\varphi_1, \ldots, \varphi_n)$ on U determine a basis $\det d\varphi := d\varphi_1 \wedge \cdots \wedge d\varphi_n$ of $\det T^*X|_U$, and a local trivialization $\psi = (\psi_1, \ldots, \psi_n)$ of $V|_U$ determines a basis $\det \psi := \psi_1 \wedge \cdots \wedge \psi_n$ of $\det V|_U$.

Let (L, ρ) be a relative orientation of $V \to X$. A local trivialization ψ is called *compatible* with Nisnevich coordinates (U, φ) and the relative orientation (L, ρ) if

$$\det \psi \otimes \det d\varphi \in \det V|_U \otimes \det T^*X|_U \cong \mathrm{Hom}(\det TX|_U, \det V|_U)$$

is a square.

Since Nisnevich coordinates always exist for $n \geqslant 1$, it is natural to wonder whether a compatible local trivialization always exists. Of course, if $V \to X$ is not relatively orientable, then no compatible local trivializations can exist, but this turns out to be the only obstruction.

Proposition 2.7. *Let X be a k-scheme of dimension n. Let $V \to X$ be a relatively orientable vector bundle of rank n, and let $\rho : \det V \otimes \omega_X \xrightarrow{\cong} L^{\otimes 2}$ be a relative* *orientation of $V \to X$. Given Nisnevich coordinates $\varphi : U \to \mathbb{A}^n_k$ on an open subscheme $U \subseteq X$, there exists a local trivialization $\psi : V|_U \to \mathbb{A}^n_k \times U \to \mathbb{A}^n_k \to \mathbb{A}^n_k$ that is compatible with (U, φ) and (L, ρ).*

Proof. The Nisnevich coordinates $\varphi = (\varphi_1, \ldots, \varphi_n)$ on U determine a local trivialization $d\varphi := (d\varphi_1, \ldots, d\varphi_n)$ on the cotangent bundle $T^*X|_U$, which in turn determines the distinguished basis element $\det d\varphi := d\varphi_1 \wedge \cdots \wedge d\varphi_n$ of $\omega_X|_U := \det T^*X|_U$ (considered as a rank one $\mathcal{O}_X(U)$-module). Let $\psi = (\psi_1, \ldots, \psi_n) : V|_U \to \mathbb{A}^n_k \times U \to \mathbb{A}^n_k$ be a local trivialization. This determines the distinguished basis element $\det \psi := \psi_1 \wedge \cdots \wedge \psi_n$ of $\det V|_U$ (considered as a rank one $\mathcal{O}_X(U)$-module).

If $\rho(\det \psi \otimes \det d\varphi) = \ell \otimes \ell$ for some $\ell \in L|_U$, then we are done. Otherwise, note that $L|_U^{\otimes 2} \cong L|_U \cong \mathcal{O}_X(U)$ and hence $\rho(\det \psi \otimes \det d\varphi) \in L|_U^{\otimes 2} \cong \mathcal{O}_X(U)$. Let $f \in \mathcal{O}_X(U)$ be the image of $\rho(\det \psi \otimes \det d\phi)$, and let $\psi' = (f\psi_1, \psi_2, \ldots, \psi_n)$. Now the image in $\mathcal{O}_X(U)$ of $\rho(\det \psi' \otimes \det d\phi)$ is f^2. Letting $\ell \in L|_U$ be the preimage of f under $L|_U \cong \mathcal{O}_X(U)$, we have $\rho(\det \psi' \otimes \det d\varphi) = \ell \otimes \ell$. □

We can now define the local index of a section $\sigma : X \to V$ at a zero $p \in X$ [37, Definition 30].

Definition 2.8. Let X be a k-scheme of dimension $n \geqslant 1$, and let $V \to X$ be a relatively orientable vector bundle. Fix a relative orientation of V. The *local index* $\mathrm{ind}_p \, \sigma$ of a section $\sigma : X \to V$ at an isolated zero $p \in X$ is defined to be the local $\mathbb{A}^1$-degree $\deg_p(\psi \circ \sigma \circ \varphi^{-1}) \in \mathrm{GW}(k)$, where (U, φ) are Nisnevich coordinates about p and ψ is a local trivialization that is compatible with (U, φ) and the chosen relative orientation of V. By [37, Corollary 31], the local index is independent of the choice of Nisnevich coordinates and compatible local trivialization.

Remark 2.9. Kass and Wickelgren originally defined the local index in terms of the Scheja–Storch form [64], which was known to be equal to the local $\mathbb{A}^1$-degree under certain hypotheses [36]. It was later shown by Bachmann and Wickelgren that these hypotheses can be removed [2, Section 7].

2.1.1 Local $\mathbb{A}^1$-degree

Since the local index of a section is defined as a local $\mathbb{A}^1$-degree, we will briefly recall the relevant definitions. We will then provide a survey of a few techniques for computing local $\mathbb{A}^1$-degrees.

Let $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ be a morphism with an isolated zero p. This induces a map $\bar{f} : \mathbb{A}^n_k/(\mathbb{A}^n_k - \{p\}) \to \mathbb{A}^n_k/(\mathbb{A}^n_k - \{0\})$ of motivic spaces. By excision, we have $\mathbb{A}^n_k/(\mathbb{A}^n_k - \{q\}) \simeq \mathbb{P}^n_k/(\mathbb{P}^n_k - \{q\})$ for any closed point q. If q is k-rational (as is the case for 0), then Morel and Voevodsky's purity theorem implies that $\mathbb{P}^n_k/(\mathbb{P}^n_k - \{q\}) \simeq \mathbb{P}^n_k/\mathbb{P}^{n-1}_k$. Since p need not be k-rational, we instead use the collapse map $c_p : \mathbb{P}^n_k/\mathbb{P}^{n-1}_k \to \mathbb{P}^n_k/(\mathbb{P}^n_k - \{p\})$. Putting this all together enables us to define the local $\mathbb{A}^1$-degree.

Definition 2.10. The *local* $\mathbb{A}^1$*-degree* of $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ at an isolated zero p is the $\mathbb{A}^1$-degree of the composite

$$\mathbb{P}^n_k/\mathbb{P}^{n-1}_k \xrightarrow{c_p} \mathbb{P}^n_k/(\mathbb{P}^n_k - \{p\}) \simeq \mathbb{A}^n_k/(\mathbb{A}^n_k - \{p\}) \xrightarrow{\bar{f}} \mathbb{A}^n_k/(\mathbb{A}^n_k - \{0\}) \simeq \mathbb{P}^n_k/\mathbb{P}^{n-1}_k.$$

In an influential paper, Kass and Wickelgren showed that the local $\mathbb{A}^1$-degree can be computed in terms of commutative algebra (via the *Eisenbud–Khimshiashvili–Levine* or *EKL form*), provided that p is k-rational or f is étale at p [36]. In joint work with Brazelton, Burklund, Montoro, and Opie, we weaken the assumption on p by proving an analogous result whenever $k(p)/k$ is a finite separable extension [9].

Construction 2.11 (EKL form)**.** Let $f = (f_1, \ldots, f_n) : \mathbb{A}^n_k \to \mathbb{A}^n_k$ have an isolated k-rational zero at p, and let $\mathfrak{m}_p = (x_1 - p_1, \ldots, x_n - p_n)$ be the maximal ideal in $k[x_1, \ldots, x_n]$ corresponding to p. The *EKL form* is a bilinear form on the local ring $Q_p := \frac{k[x_1,\ldots,x_n]_{\mathfrak{m}_p}}{(f_1,\ldots,f_n)}$, which is a finite k-algebra by the assumption that p is an isolated zero. Since f vanishes at p, we can write

$$f_i = \sum_{j=1}^{n} a_{ij}(x_j - p_j)$$

for each $1 \leqslant i \leqslant n$. While such a composition is not unique, Scheja and Storch showed that the image E of $\det(a_{ij})$ in Q_p is independent of the choices of a_{ij}; moreover, E generates the *socle* of Q_p, which is the annihilator of the maximal ideal $\mathfrak{m}_p$ [64]. Scheja and Storch then construct an explicit k-linear form $\eta : Q_p \to k$ that satisfies $\eta(E) = 1$. From this, they obtain a symmetric non-degenerate bilinear form $\Phi_\eta : Q_p \times Q_p \to k$ given by $\Phi_\eta(x, y) = \eta(xy)$.

The insight of Eisenbud–Levine [27] and Khimshiashvili [34] is that if $\lambda : Q_p \to k$ is any k-linear form satisfying $\lambda(E) = 1$, then Φ_λ is isomorphic to Φ_η. An *EKL form* is a bilinear form Φ_λ for any such λ. By [36], the local $\mathbb{A}^1$-degree can therefore be computed at a k-rational point by:

1. Calculating a k-basis for Q_p,
2. Calculating $E \in Q_p$,
3. Picking a linear form $\lambda : Q_p \to k$ such that $\lambda(E) = 1$.
4. Taking the isomorphism class of Φ_λ.

If p is not k-rational but $k(p)/k$ is separable, then one can compute an EKL form $\Phi_{\lambda'}$ of the base change $f_{k(p)}$ of f at the $k(p)$-rational lift of p. The local $\mathbb{A}^1$-degree will be given by $\deg_p(f) = \mathrm{Tr}_{k(p)/k}\, \beta$, where β is the isomorphism class of $\Phi_{\lambda'}$ [9].

Remark 2.12. If the EKL form of f at a separable point p is rank $[k(p) : k]$, then the EKL form (and thus the local $\mathbb{A}^1$-degree) is given by the Jacobian: $\deg_p(f) = \mathrm{Tr}_{k(p)/k}\langle \mathrm{Jac}(f)(p)\rangle$ [36, Proposition 15] (see also [64, (4.7) Korollar]). Since the Jacobian of a morphism has a straightforward geometric interpretation (as discussed in Section 3.4), it is generally desirable to reduce to this "transverse" case. We describe in Section 2.1.2 the *dynamic local* $\mathbb{A}^1$*-degree*, which allows one to make such reductions.

While computing the local $\mathbb{A}^1$-degree via EKL forms is useful, there are limitations to this approach. For example, if $k(p)/k$ is not a separable extension, it is not clear how to use the EKL form to compute the local $\mathbb{A}^1$-degree (except in dimension 1 [10]). Another limitation is that the EKL form can only be used to compute a *local* degree, instead of the *global* $\mathbb{A}^1$*-degree*

$$\deg(f) := \sum_{p\in f^{-1}(0)} \deg_p(f).$$

In dimension 1, Cazanave gives a method for computing the global $\mathbb{A}^1$-degree in terms of the *Bézoutian* [16]. In joint work with Brazelton and Pauli, we proved that the multivariate Bézoutian can be used to compute local and global $\mathbb{A}^1$-degrees in any dimension, with no restrictions on p in the local case [12].

Construction 2.13 (Bézoutian)**.** Consider the polynomials

$$\Delta_{ij} := \frac{f_i(Y_1,\dots,Y_{j-1},X_j,\dots,X_n) - f_i(Y_1,\dots,Y_j,X_{j+1},\dots,X_n)}{X_j - Y_j}$$

in $k[X_1\dots,X_n,Y_1,\dots,Y_n]$. The *(multivariate) Bézoutian* of f is the determinant $\text{Béz}(f) := \det(\Delta_{ij})$. Let $\boldsymbol{X} = (X_1,\dots,X_n)$ and $\boldsymbol{Y} = (Y_1,\dots,Y_n)$. We define $\overline{\text{Béz}}(f)$ to be the image of $\text{Béz}(f)$ in $\frac{k[\boldsymbol{X},\boldsymbol{Y}]}{(f_1(\boldsymbol{X}),f_1(\boldsymbol{Y}),\dots,f_n(\boldsymbol{X}),f_n(\boldsymbol{Y}))}$. This ring is isomorphic to $\frac{k[\boldsymbol{x}]}{(f_1(\boldsymbol{x}),\dots,f_n(\boldsymbol{x}))}\otimes\frac{k[\boldsymbol{x}]}{(f_1(\boldsymbol{x}),\dots,f_n(\boldsymbol{x}))}$, where $\boldsymbol{x} = (x_1,\dots,x_n)$. At a point p corresponding to the maximal ideal $\mathfrak{m}_p \subset k[\boldsymbol{x}]$, we define $\overline{\text{Béz}}_p(f)$ to be the image of $\text{Béz}(f)$ in $\frac{k[\boldsymbol{x}]_{\mathfrak{m}_p}}{(f_1(\boldsymbol{x}),\dots,f_n(\boldsymbol{x}))}\otimes\frac{k[\boldsymbol{x}]_{\mathfrak{m}_p}}{(f_1(\boldsymbol{x}),\dots,f_n(\boldsymbol{x}))}$. Let $\{b_1(\boldsymbol{x}),\dots,b_m(\boldsymbol{x})\}$ be a k-basis for $\frac{k[\boldsymbol{x}]}{(f_1,\dots,f_n)}$ (respectively, $\frac{k[\boldsymbol{x}]_{\mathfrak{m}_p}}{(f_1,\dots,f_n)}$). Write $\overline{\text{Béz}}(f)$ (respectively, $\overline{\text{Béz}}_p(f)$) as $\sum_{i,j=1}^m a_{ij}b_i(\boldsymbol{x})\otimes b_j(\boldsymbol{x})$

for some $a_{ij} \in k$. Then $\deg(f)$ (respectively, $\deg_p(f)$) is given by the isomorphism class of the bilinear form represented by the Gram matrix (a_{ij}) [12]. This gives a straightforward algebraic characterization of the $\mathbb{A}^1$-degree, and this method has been implemented in Sage [11].

Remark 2.14. Recall that $\deg : [\mathbb{P}^1_k, \mathbb{P}^1_k] \to \mathrm{GW}(k)$ is merely a surjection, not an isomorphism. Morel further showed that $[\mathbb{P}^1_k, \mathbb{P}^1_k] \cong \mathrm{GW}(k) \times_{k^\times/(k^\times)^2} k^\times$ [51]. In the global, dimension 1 case, Cazanave shows that the extra $k^\times$ factor is accounted for by the determinant of the Bézoutian [16]. That is, denoting the isomorphism class of the Bézoutian bilinear form by Béz, the *unstable* $\mathbb{A}^1$*-degree*

$$\deg^u : [\mathbb{P}^1_k, \mathbb{P}^1_k] \to \mathrm{GW}(k) \times_{k^\times/(k^\times)^2} k^\times$$

is given by $(\text{Béz}, \det \text{Béz})$.

2.1.2 Dynamic local $\mathbb{A}^1$*-degree*

Using dynamic intersections, one can relate a special intersection to a generic one [31, Section 11]. The classical dynamic intersection was enriched by Pauli to give a dynamic $\mathbb{A}^1$-Euler number [55], as well as by Pauli–Wickelgren to give a dynamic $\mathbb{A}^1$-Milnor number [56]. In fact, Pauli–Wickelgren's approach can be repeated almost verbatim to give a dynamic local $\mathbb{A}^1$-degree for any $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$. We recall the details here, with [56, Section 6.3] as the standing reference for Section 2.1.2. This will culminate in Theorem 2.26, which is essentially a rephrasing of [56, Theorem 5]. Throughout this section, let k be a field with $\operatorname{char} k \neq 2$. We begin with the classical computation of $\mathrm{GW}(k[\![t]\!])$:

Proposition 2.15. *The map* $\langle r(t)\rangle \mapsto \langle r(0)\rangle$ *defines an isomorphism*

$$\mathrm{ev}_0 : \mathrm{GW}(k[\![t]\!]) \xrightarrow{\cong} \mathrm{GW}(k)$$

with inverse induced by the inclusion map $k \hookrightarrow k[\![t]\!]$.

Proof. This follows from the fact that $\mathrm{GW}(k[\![t]\!])$ is generated by elements of the form $\langle r(t)\rangle$ for units $r(t) \in k[\![t]\!]^\times$ (see e.g. [2, Lemma B.3]). Any such unit satisfies $r(0) \neq 0$, so we may write $r(t) = \sum_{i=0}^\infty a_i t^i = a_0(1 + \sum_{i=1}^\infty \frac{a_i}{a_0} t^i)$. Since $\operatorname{char} k \neq 2$, there exists a square root $s(t) \in k[\![t]\!]$ of $1 + \sum_{i=1}^\infty \frac{a_i}{a_0} t^i$. In particular, in $\mathrm{GW}(k[\![t]\!])$, we have

$$\begin{aligned}\langle r(t)\rangle &= \langle a_0 s(t)^2\rangle \\ &= \langle a_0\rangle = \langle r(0)\rangle.\end{aligned}$$

□

We now summarize the relationships between $\mathrm{GW}(k)$, $\mathrm{GW}(k[\![t]\!])$, and $\mathrm{GW}(k(\!(t)\!))$. Any element of $k(\!(t)\!)$ is either of the form u or ut, where u is a unit under the (t)-adic valuation. The second residue homomorphism $\partial_t : \mathrm{GW}(k(\!(t)\!)) \to \mathrm{W}(k)$ is defined by $\partial_t\langle ut\rangle = \langle \bar{u}\rangle$ and $\partial_t\langle u\rangle = 0$, where $\bar{u} \in k$ is the residue of u in $k(\!(t)\!)/(t)$. More generally, the second residue homomorphism is defined on Milnor–Witt K-groups

$$\partial_t : \mathrm{K}_n^{\mathrm{MW}}(k(\!(t)\!)) \to \mathrm{K}_{n-1}^{\mathrm{MW}}(k),$$

with $\mathrm{K}_n^{\mathrm{MW}}(k[\![t]\!]) := \ker \partial_t$ [52, p. 58]. Setting $n = 0$ recovers the short exact sequence

$$0 \to \mathrm{GW}(k[\![t]\!]) \to \mathrm{GW}(k(\!(t)\!)) \xrightarrow{\partial_t} \mathrm{W}(k) \to 0$$

of abelian groups. Denote the inclusion of $\ker \partial_t$ by $\imath : \mathrm{GW}(k[\![t]\!]) \to \mathrm{GW}(k(\!(t)\!))$. Let $\pi : \mathrm{GW}(k(\!(t)\!)) \twoheadrightarrow \mathrm{GW}(k)$ be projection on the first factor under Springer's theorem [39, Chapter VI, Theorem 1.4]. Together with $\mathrm{ev}_0 : \mathrm{GW}(k[\![t]\!]) \to \mathrm{GW}(k)$, these maps form a commutative triangle.

Proposition 2.16. *The following diagram commutes.*

$$\begin{tikzcd} \mathrm{GW}(k[\![t]\!]) \arrow[r, hook, "\imath"] \arrow[d, "\mathrm{ev}_0"', "\cong"] & \mathrm{GW}(k(\!(t)\!)) \arrow[dl, two heads, "\pi"] \\ \mathrm{GW}(k) & \end{tikzcd}$$

Proof. The composite $\imath \circ \mathrm{ev}_0^{-1} : \mathrm{GW}(k) \hookrightarrow \mathrm{GW}(k(\!(t)\!))$ is the injection induced by the inclusion $k \hookrightarrow k(\!(t)\!)$ defined by $a \mapsto a$ [39, p. 146], so $\pi \circ \imath = \mathrm{ev}_0$ by the definition of π. $\square$

Given a map $\mathbb{A}^n_k \to \mathbb{A}^n_k$, we will build a deformation (that is, a map over $k[\![t]\!]$) whose local degree at a special fiber is our local degree valued in $\mathrm{GW}(k)$. The general fiber of this map will have local degree valued in $\mathrm{GW}(k(\!(t)\!))$. Proposition 2.16 will enable us to relate these two local degrees and exploit the genericity of transverse intersections.

Let $f_1, \ldots, f_n \in k[x_1, \ldots, x_n]$. Given $g_1, \ldots, g_n \in k[\![t]\!][x_1, \ldots, x_n]$, let

$$X := \mathbb{V}(f_1 + tg_1, \ldots, f_n + tg_n) \subseteq \mathbb{A}^n_{k[\![t]\!]}.$$

Notation 2.17. Given a scheme $Y \to \operatorname{Spec} k[\![t]\!]$, denote its special fiber by $Y_0 := \operatorname{Spec} k \times_{\operatorname{Spec} k[\![t]\!]} Y$ and its generic fiber by $Y_t := \operatorname{Spec} k(\!(t)\!) \times_{\operatorname{Spec} k[\![t]\!]} Y$.

Note that $\mathbb{V}(f_1, \ldots, f_n) = X_0$. Since $k[\![t]\!]$ is a local ring, [63, Lemma 04GG (12)] implies that $X = X^{\mathrm{fin}} \amalg X^{\geqslant 1}$, where $X^{\mathrm{fin}} \to \operatorname{Spec} k[\![t]\!]$ is finite and $(X^{\geqslant 1})_0$ is a union of irreducible k-schemes of dimension at least 1.

Notation 2.18. Given a closed point $p \in X_0$, let X^p be the union of all irreducible components of X containing p.

The scheme X^p is a finite collection of points, namely p and the points that p splits into in the generic fiber X_t. To see this, we first show that $X^p \to \operatorname{Spec} k[\![t]\!]$ is finite.

Proposition 2.19. *If $p \in X_0$ is isolated, then $X^p \to \operatorname{Spec} k[\![t]\!]$ is finite.*

Proof. Since p is isolated, the local ring $\mathcal{O}_{X_0,p}$ is finite as a k-module. In particular, the special fiber of any irreducible component of X containing p must be finite over k, so the decomposition $X = X^{\mathrm{fin}} \amalg X^{\geqslant 1}$ implies that X^p is a closed subscheme of

X^{fin}. The finiteness of $X^p \to \operatorname{Spec} k[\![t]\!]$ now follows from the finiteness of $X^{\text{fin}} \to \operatorname{Spec} k[\![t]\!]$. $\square$

We are now ready to show that p is the only point in the special fiber $(X^p)_0$. Since $Y = Y_0 \amalg Y_t$ (set-theoretically) for any $k[\![t]\!]$-scheme Y, it will follow that $X^p - \{p\} = (X^p)_t$. By construction, $(X^p)_t$ consists of the points that map to p under $X_t \to X_0$, or in other words, the points that $p \in X_0$ splits into in the generic fiber X_t.

Proposition 2.20. *If $p \in X_0$ is isolated, then p is the only point of $(X^p)_0$.*

Proof. Let $x_t \in (X^p)_t$ be a point. The residue field $\kappa(x_t)$ of x_t is a finite extension of $k(\!(t)\!)$. Letting R be the integral closure of $k[\![t]\!]$ in $\kappa(x_t)$, we get a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} \kappa(x_t) & \longrightarrow & X^p \\ \downarrow & & \downarrow \\ \operatorname{Spec} R & \longrightarrow & \operatorname{Spec} k[\![t]\!]. \end{array} \tag{2.2}$$

The map $\operatorname{Spec} R \to \operatorname{Spec} k[\![t]\!]$ is finite by [63, Lemma 032Q] and [63, Lemma 032L] if $\operatorname{char} k = 0$ or [63, Lemma 032N] if $\operatorname{char} k \neq 0$. Since $X^p \to \operatorname{Spec} k[\![t]\!]$ is finite, it is also a proper morphism, so the valuative criterion for properness [63, Lemma 0A40] implies that there is a unique morphism $\operatorname{Spec} R \to X^p$ that commutes with Diagram 2.2. Moreover, the image of $\operatorname{Spec} R$ (which we denote by x) is a component of X^p, since having a finite map $X^p \to \operatorname{Spec} k[\![t]\!]$ implies that $\dim X^p \leqslant \dim \operatorname{Spec} k[\![t]\!] = \dim R$ [63, Lemma 0ECG]. Thus $p \in x$ by definition of X^p.

By [7, Section 3.2.4, Theorem 2], R is a complete discrete valuation ring. Since $k[\![t]\!] \to R$ is finite, $\dim R = 1$. The Cohen structure theorem (see e.g. [63, Section 0323]) thus implies that $R \cong L[\![u]\!]$ for some finite extension L/k and some parameter

u. In particular, the special fiber $(\operatorname{Spec} R)_0$ contains a unique point, so the special fiber $x_0 \in (X^p)_0$ consists of a unique point. As $p \in x$, it follows that $x_0 = p$, so the special fiber of any component of X^p consists solely of the point p. $\square$

Our next goal is to show that $\mathcal{O}_{X^p}(X^p)$ is a free $k[\![t]\!]$-module and that $(f_1 + tg_1, \ldots, f_n + tg_n)$ is regular sequence. This will allow us to define the local degree $\deg_p^{\mathbb{A}^1}(f_1 + tg_1, \ldots, f_n + tg_n) \in \mathrm{GW}(k[\![t]\!])$ as the isomorphism class of the Scheja–Storch bilinear form $\mathcal{O}_{X^p}(X^p) \times \mathcal{O}_{X^p}(X^p) \to k[\![t]\!]$ [64, §3].

Proposition 2.21. *If $p \in X_0$ is isolated, then $\mathcal{O}_{X^p}(X^p)$ is a local ring.*

Proof. In the proof of Proposition 2.20, we saw that (as a set of points) X^p consists of a set of maximal points $x_t \in (X^p)_t$ and a unique closed point $p \in (X^p)_0$. Since $X^p \to \operatorname{Spec} k[\![t]\!]$ is finite, it is also quasi-compact. In particular, the topological space underlying X^p is quasi-compact by [63, Lemma 01K4]. Now [17, Proposition 4] implies that $X^p = \operatorname{Spec} R$ for some local ring R. In particular, X^p is affine, so $R \cong \mathcal{O}_{X^p}(X^p)$. $\square$

Proposition 2.22. *If $p \in X_0$ is isolated, then there exists a $k[\![t]\!]$-module M such that $(f_1 + tg_1, \ldots, f_n + tg_n)$ is a regular sequence in M and $\mathcal{O}_{X^p}(X^p) \cong M/(f_1 + tg_1, \ldots, f_n + tg_n)$.*

Proof. Let $\mathfrak{m} \subset k[\![t]\!][x_1, \ldots, x_n]$ be the maximal ideal corresponding to the point p over $k[\![t]\!]$ (with $t \in \mathfrak{m}$), and let $\mathfrak{m}_0 = \mathfrak{m}/(t) = \mathfrak{m} \cap k[x_1, \ldots, x_n]$ (which corresponds to p over k). Set $R = \frac{k[\![t]\!][x_1,\ldots,x_n]}{(f_1+tg_1,\ldots,f_n+tg_n)}$. Let $\min(R)$ be the set of minimal primes of R (corresponding to the irreducible components of X), and let $\min(R)^p$ be the set of minimal primes of R that are contained in the image of $\mathfrak{m}$ (corresponding to the irreducible components of X^p). Finally, let $S = R - \min(R)^p$. We claim that $\mathcal{O}_{X^p}(X^p) \cong S^{-1}R$, from which it will follow that $\mathcal{O}_{X^p}(X^p) \cong \frac{Q^{-1}(k[\![t]\!][x_1,\ldots,x_n])}{(f_1+tg_1,\ldots,f_n+tg_n)}$ for some multiplicatively closed subset $Q \subset k[\![t]\!][x_1, \ldots, x_n]$ (since localization commutes with quotients).

To prove the claim, we first note that S is multiplicatively closed. Indeed, since X^p is finite over a Noetherian base, X^p is Noetherian. Thus X^p has finitely many components, so $\min(R)^p$ is a finite set of primes. Moreover, $\mathfrak{p} \in \operatorname{Spec} S^{-1}R$ if and only if $\mathfrak{p} \in \min(R)^p$, so $\operatorname{Spec} S^{-1}R = X^p$.

The assumption that $p \in X_0$ is isolated implies that the local ring $\mathcal{O}_{X_0,p}$ has dimension 0. Note that $\mathcal{O}_{X_0,p} \cong \frac{Q^{-1}(k[\![t]\!][x_1,\ldots,x_n])}{(f_1+tg_1,\ldots,f_n+tg_n,t)}$. Since $M := Q^{-1}(k[\![t]\!][x_1, \ldots, x_n])_{\mathfrak{m}}$ is a regular local ring (of dimension $n+1$), it is a local Cohen–Macaulay ring. Thus [63, Lemma 02NJ] implies that $(f_1 + tg_1, \ldots, f_n + tg_n, t)$ is a regular sequence in M. It follows that $(f_1 + tg_1, \ldots, f_n + tg_n)$ is also a regular sequence in M. Since $S^{-1}R$ is already local by Proposition 2.21, we also have $\mathcal{O}_{X^p}(X^p) \cong S^{-1}R \cong M/(f_1 + tg_1, \ldots, f_n + tg_n)$. □

Proposition 2.23. *If $p \in X_0$ is isolated, then $X^p \to \operatorname{Spec} k[\![t]\!]$ is flat.*

Proof. Using the notation in the proof of Proposition 2.22, we have that $k[\![t]\!]$ is a regular local ring of dimension 1, $S^{-1}R$ is Cohen–Macaulay of dimension 1, and $S^{-1}R \otimes k \cong \mathcal{O}_{X_0,p}$ has dimension 0. Thus [44, Theorem 23.1 (p. 179)] implies that $X^p \to \operatorname{Spec} k[\![t]\!]$ is flat. □

Proposition 2.24. *If $p \in X_0$ is isolated, then $\mathcal{O}_{X^p}(X^p)$ is a free $k[\![t]\!]$-module.*

Proof. Since $k[\![t]\!]$ is Noetherian and $X^p \to \operatorname{Spec} k[\![t]\!]$ is finite, flatness (Proposition 2.23) implies that $\mathcal{O}_{X^p}(X^p)$ is a projective $k[\![t]\!]$-module. Since projective modules are locally free, $\mathcal{O}_{X^p}(X^p)$ being local (Proposition 2.21) that $\mathcal{O}_{X^p}(X^p)$ is a free $k[\![t]\!]$-module. □

We can now define $\deg_p^{\mathbb{A}^1}(f_1 + tg_1, \ldots, f_n + tg_n) \in \mathrm{GW}(k[\![t]\!])$.

Definition 2.25. Let $(f_1, \ldots, f_n) : \mathbb{A}^n_k \to \mathbb{A}^n_k$ with isolated zero p. Let $g_1, \ldots, g_n$ be any elements of $k[\![t]\!][x_1, \ldots, x_n]$ such that

$$\operatorname{Spec} \frac{k[\![t]\!][x_1, \ldots, x_n]}{(f_1 + tg_1, \ldots, f_n + tg_n)} \to \operatorname{Spec} k[\![t]\!]$$

is finite and flat. Let $X = \mathbb{V}(f_1 + t_1 g_1, \ldots, f_n + tg_n)$. Define $\deg_p^{\mathbb{A}^1}(f_1 + tg_1, \ldots, f_n + tg_n) \in \mathrm{GW}(k[\![t]\!])$ to be the isomorphism class of the Scheja–Storch bilinear pairing $\mathcal{O}_{X^p}(X^p) \times \mathcal{O}_{X^p}(X^p) \to k[\![t]\!]$ determined by the regular sequence $(f_1 + tg_1, \ldots, f_n + tg_n)$.

Putting this all together, we get the following rephrasing of [56, Theorem 5]:

Theorem 2.26 (Dynamic local $\mathbb{A}^1$-degree). *Let $(f_1, \ldots, f_n) : \mathbb{A}^n_k \to \mathbb{A}^n_k$ with isolated zero p. Let $g_1, \ldots, g_n$ be any elements of $k[\![t]\!][x_1, \ldots, x_n]$ such that*

$$\operatorname{Spec} \frac{k[\![t]\!][x_1, \ldots, x_n]}{(f_1 + tg_1, \ldots, f_n + tg_n)} \to \operatorname{Spec} k[\![t]\!]$$

is finite and flat. Let $X = \mathbb{V}(f_1 + t_1 g_1, \ldots, f_n + tg_n)$, and let $X_t^p := (X^p)_t \subset \mathbb{A}^n_{k(\!(t)\!)}$ be the collection of points that p splits into under the deformation $X_0 \mapsto X_t$. Then

$$\deg_p^{\mathbb{A}^1}(f_1, \ldots, f_n) = \pi|_{\mathrm{im}(\imath)} \left(\sum_{z \in X_t^p} \deg_z^{\mathbb{A}^1}(f_1 + tg_1, \ldots, f_n + tg_n) \right)$$

as elements of $\mathrm{GW}(k)$.

Proof. Let $f = (f_1, \ldots, f_n)$ and $g = (g_1, \ldots, g_n)$. By construction, we have

$$\mathrm{ev}_0(\deg_p^{\mathbb{A}^1}(f + tg)) = \deg_p^{\mathbb{A}^1}(f)$$

as elements of $\mathrm{GW}(k)$. The map $\mathbb{A}^n_{k[\![t]\!]} \to \mathbb{A}^n_{k(\!(t)\!)}$ induced by the inclusion $k[\![t]\!] \hookrightarrow k(\!(t)\!)$ sends $p \in X \subset \mathbb{A}^n_{k[\![t]\!]}$ to $X_t^p \subset \mathbb{A}^n_{k(\!(t)\!)}$, so

$$\imath(\deg_p^{\mathbb{A}^1}(f + tg)) = \sum_{z \in X_t^p} \deg_z^{\mathbb{A}^1}(f + tg)$$

as elements of $\mathrm{GW}(k((t)))$. The result now follows from Proposition 2.16. $\square$

In Chapter 5, we will use Theorem 2.26 to give an enrichment of Bézout's theorem without the transversality hypothesis.

2.1.3 *Computing the local degree in families*

In essence, the dynamic approach enables us to compute the local $\mathbb{A}^1$-degree at a point by computing a sum of local $\mathbb{A}^1$-degrees over a nearby fiber. Using Harder's theorem [36, Lemma 30], we might instead try to compute $\deg_p^{\mathbb{A}^1}(f)$ by computing a sum of local $\mathbb{A}^1$-degrees over an arbitrary fiber in a family containing p. Since $\mathbb{V}(f)$ is zero dimensional, a family $X \to \operatorname{Spec} k[t]$ with special fiber $X_0 = \mathbb{V}(f)$ is a branched cover of the affine line. We want to separate p, a point of higher intersection multiplicity, into a set of reduced points. We then wish to express $\deg_p^{\mathbb{A}^1}(f)$ as a sum of local $\mathbb{A}^1$-degrees over this set of reduced points.

However, if X is ramified somewhere between the special fiber X_0 and the fiber over which we wish to compute the local $\mathbb{A}^1$-degree, then we may lose track of the individual points at which to compute — there can be multiple points in the fiber X_0 that belong to the same connected component of X (see Figure 2.1). We will avoid this issue by assuming that X is ramified only at the fiber containing p. We can then remove the unwanted components of X by localizing to the irreducible components of X that contain p (see Figure 2.2). In a sense, our ramification assumption allows us to mimic the dynamic approach over the non-local base $\operatorname{Spec} k[t]$.

Before describing the familial local $\mathbb{A}^1$-degree (Theorem 2.28), we need the following analog of Proposition 2.19:

Lemma 2.27. *Let $\varphi : X \to \operatorname{Spec} k[t]$ be a morphism of finite type, where X is affine. Assume that every irreducible component of X surjects onto $\operatorname{Spec} k[t]$ under φ, that φ is unramified away from $t = 0$, and that X_0 is a single closed point. Then φ is*

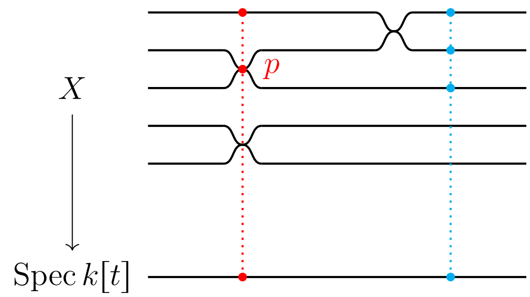

FIGURE 2.1: Losing track of points that split off from p

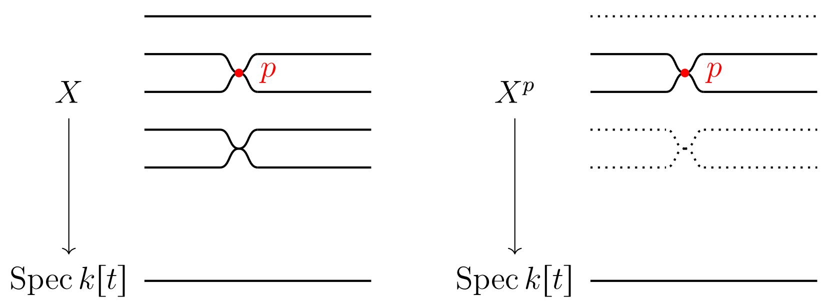

FIGURE 2.2: Removing disjoint sheets

finite and flat.

Proof. We will first show that φ is flat. Write $X = \operatorname{Spec} A$ for some $k[t]$-module A. Since $k[t]$ is a Dedekind domain, it suffices to show that A is torsion-free. Suppose $g \in k[t]$ is a non-zero element that annihilates some $a \in A$. Then for any irreducible component $Y \subseteq X$ on which a does not vanish, we have $\varphi(Y) \subseteq \mathbb{V}(g) \subsetneq \operatorname{Spec} k[t]$. But this contradicts our assumption that each irreducible component of X surjects onto $\operatorname{Spec} k[t]$, so we deduce that φ is flat.

Next, we show that φ has finite fibers. Since φ is affine and finite type, φ is quasi-finite if and only if it has finite fibers [63, Lemma 02NH]; the same is also true for the restriction of φ to $\varphi' : X - X_0 \to \mathbb{A}^1_k - \{0\}$. The map φ' is unramified by assumption and is therefore locally quasi-finite by [63, Lemma 02V5]. Since φ' is affine and hence quasi-compact [63, Lemma 01S7], it follows that φ' is quasi-finite [63, Lemma 01TJ]. Thus φ' has finite fibers. The fiber of φ above 0 is finite by assumption, so φ has finite fibers.

Finally, note that if Z is an irreducible component of $X - X_0$, then each fiber of Z consists of a single point. Indeed, Z is connected (being irreducible), so if some fiber of Z consists of more than one point, then Z consists of more than one sheet. But $Z \to \mathbb{A}^1_k - \{0\}$ is unramified, so these sheets must remain disjoint. This contradicts the assumption that Z is irreducible. Thus $Z \to \mathbb{A}^1_k - \{0\}$ is injective, so this map is an isomorphism. It follows that $Z \cup X_0$ is isomorphic to $\operatorname{Spec} k[t]$, so X is a finite union of isomorphic copies of $\operatorname{Spec} k[t]$. As a result, X is finite over $\operatorname{Spec} k[t]$. $\square$

Theorem 2.28 (Familial local $\mathbb{A}^1$-degree)**.** *Let $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ with isolated zero p such that $k(p)/k$ separable. Let $F : \mathbb{A}^n_{k[t]} \to \mathbb{A}^n_{k[t]}$ be a morphism such that $\mathbb{V}(F) \to$* $\operatorname{Spec} k[t]$ *is flat and $F|_{t=0} = f$. Assume that $\mathbb{V}(F)$ is unramified away from $t = 0$. Then for any closed point $c \in \mathbb{A}^1_k$, the perturbation $\tilde{f} := F|_{t=c} : \mathbb{A}^n_k \to \mathbb{A}^n_k$ of f has a set of zeros $Z \subseteq \tilde{f}^{-1}(0)$ such that*

$$\deg_p^{\mathbb{A}^1}(f) = \sum_{q \in Z} \deg_q^{\mathbb{A}^1}(\tilde{f}).$$

Proof. We will construct a pair (Q, β), where Q is a finite locally free $k[t]$-module and β is a non-degenerate symmetric bilinear form on Q, such that

(i) the isomorphism class of $\beta|_{t=0}$ is $\deg_p^{\mathbb{A}^1}(f)$, and

(ii) the isomorphism class of $\beta|_{t=c}$ is $\sum_{q \in Z} \deg_q^{\mathbb{A}^1}(\tilde{f})$.

Once we have constructed (Q, β), it will follow from [36, Lemma 30] that $\deg_p^{\mathbb{A}^1}(f) = \sum_{q \in \tilde{f}^{-1}(0)} \deg_q^{\mathbb{A}^1}(\tilde{f})$.

Let $\mathfrak{m} \subset k[t][x_1, \ldots, x_n]$ be the maximal ideal corresponding to p with $t \in \mathfrak{m}$, and let P be the set of minimal prime ideals $(F_1, \ldots, F_n) \subseteq \mathfrak{p} \subset \mathfrak{m}$. Note that the prime ideals in P correspond to the irreducible components of $\mathbb{V}(F) = \operatorname{Spec} \frac{k[t][x_1,\ldots,x_n]}{(F_1,\ldots,F_n)}$ containing p. Since $\mathbb{V}(F) \to \operatorname{Spec} k[t]$ is unramified away from $t = 0$, this map is quasi-finite away from $t = 0$ [63, Lemma 02V5]. In particular, $\mathbb{V}(F) \to \operatorname{Spec} k[t]$ has finite fibers away from $t = 0$, so P is a finite set of prime ideals. It follows that $S = k[t][x_1, \dots, x_n] - P$ is multiplicatively closed. Set $Q = \frac{S^{-1}(k[t][x_1,\dots,x_n])}{(F_1,\dots,F_n)}$. The localization $\operatorname{Spec} Q$ is the restriction of the vanishing locus $\mathbb{V}(F)$ to the components that specialize to p at $t = 0$.

By construction, $\operatorname{Spec} Q_0 = \{p\}$, so Lemma 2.27 implies that Q is a finite $k[t]$-module. Since $\operatorname{Spec} Q \to \mathbb{V}(F)$ is flat by [63, Lemma 00HT (1)] and $\mathbb{V}(F) \to \operatorname{Spec} k[t]$ is flat by assumption, [63, Lemma 01U7] implies that Q is a flat $k[t]$-module. Since $k[t]$ is Noetherian, Q being a finite $k[t]$-module is equivalent to Q being a finitely presented $k[t]$-module, so [63, Lemma 00NX (1) and (7)] implies that Q is a finite locally free $k[t]$-module. (In fact, Q is projective over a PID, so Q is even a free $k[t]$-module.)

We thus have the desired Q. We define β to be the Scheja–Storch form on Q associated to the sequence $(F_1, \dots, F_n)$. This gives us (i), as $Q_0 = \frac{k[x_1,\dots,x_n]_\mathfrak{m}}{(f_1,\dots,f_n)}$ and $\beta|_{t=0}$ is the Scheja–Storch form on Q_0 associated to $F|_{t=0} = f$. Likewise, $\beta|_{t=c}$ is the Scheja–Storch form on Q_c associated to $\tilde{f}$. Since Q_c has finite k-dimension, it is an Artinian ring and thus has finitely many maximal ideals. Each of these maximal ideals corresponds to a zero of $\tilde{f}$. Let $Z \subset \mathbb{A}^n_k$ be the set of points corresponding to the maximal ideals of Q_c. It follows from e.g. [12, Lemma 4.7 and Theorem 5.1] that $\beta|_{t=c}$ is isomorphic to $\sum_{q \in Z} \deg_q^{\mathbb{A}^1}(\tilde{f})$, which gives us (ii). □

Remark 2.29. Similar to Theorem 2.28, Kass and Wickelgren have used Harder's theorem to study the $\mathbb{A}^1$-degree in families [36, 37]. In their work, they show (and utilize) that the sum of local $\mathbb{A}^1$-degrees over a given fiber is independent of the fiber chosen. Our approach describes how to remove other elements of the fiber over 0 in order to compute the local $\mathbb{A}^1$-degree at a single point (instead of over the whole fiber) by working in families.

2.2 Global fixed counts

The classical Poincaré–Hopf theorem calculates the Euler characteristic of a compact differentiable manifold as a sum of local indices over the vanishing locus of any vector field with only isolated zeros [35]. As a motivic analog, one can define the Euler class $e(V,\sigma)$ of a relatively orientable vector bundle $V \to X$ over a proper scheme with respect to a section $\sigma : X \to V$ with isolated zeros by

$$e(V,\sigma) := \sum_{p\in\sigma^{-1}(0)} \operatorname{ind}_p \sigma.$$

Bachmann and Wickelgren proved that this notion of Euler class is independent of σ [2, Theorem 1.1]. In particular, while the specific local contributions $\operatorname{ind}_p \sigma$ usually depend on the section σ (which represents an instance of the enumerative problem), the global sum of local contributions is fixed and independent of σ. It is thus desirable to be able to compute Euler classes in motivic homotopy theory. We give a brief survey of the literature on this subject.

If all zeros of a section $\sigma : X \to V$ lie in a single affine patch $\mathbb{A}^n \cong U \subseteq X$, then the Euler class $e(V)$ can be computed as a global $\mathbb{A}^1$-degree via the multivariate Bézoutian [12]. Frequently, $e(V)$ is a multiple of the hyperbolic form $\mathbb{H}$. This is explained by Srinivasan–Wickelgren (who build on the work of Levine and Fasel): if V has an odd rank direct summand (or e.g. V itself has odd rank), then $e(V)$ is hyperbolic [62, Proposition 12]. Levine–Raksit and Bachmann–Wickelgren construct a motivic Euler class in terms of coherent duality [2, 42], which is shown by Bachmann–Wickelgren to agree with the Poincaré–Hopf motivic Euler class.

Suppose X is smooth and proper over $\mathbb{Z}[1/2]$, and let $V_k \to X$ be the base change of a relatively orientable vector bundle to a field k of characteristic not 2. Let $e_\mathbb{C}$ and $e_\mathbb{R}$ be the complex and real Euler numbers, respectively, of $V \to X$. In a particularly interesting result, Bachmann and Wickelgren prove that there are only two options for $e(V_k)$: it is either $\frac{e_{\mathbb{C}}+e_{\mathbb{R}}}{2}\langle 1\rangle + \frac{e_{\mathbb{C}}-e_{\mathbb{R}}}{2}\langle -1\rangle$ or $\frac{e_{\mathbb{C}}+e_{\mathbb{R}}}{2}\langle 1\rangle + \frac{e_{\mathbb{C}}-e_{\mathbb{R}}}{2}\langle -1\rangle + \langle 2\rangle - \langle 1\rangle$ [2, Theorem 5.11]. This can even be sharpened with recent breakthroughs in Hermitian K-theory over $\mathbb{Z}$. Using [13], Bachmann–Wickelgren show that if X is smooth and proper over $\mathbb{Z}$, then

$$e(V_k) = \frac{e_{\mathbb{C}} + e_{\mathbb{R}}}{2}\langle 1\rangle + \frac{e_{\mathbb{C}} - e_{\mathbb{R}}}{2}\langle -1\rangle$$

for any field k, regardless of $\operatorname{char} k$. Strikingly, while the fixed count $e(V_k) \in \mathrm{GW}(k)$ carries arithmetic information relative to the field k, this data is completely governed by topological information over $\mathbb{R}$ and $\mathbb{C}$. This also illustrates that fixed counts in $\mathbb{A}^1$-enumerative geometry are often simple, in contrast with the potentially complicated nature of the local contributions comprising any given fixed count [12, 57].

Tacit in this entire discussion is the assumption that the zero locus of σ is isolated. Many interesting enumerative problems involve excess or residual intersections. The Scheja–Storch bilinear pairing was generalized by Eisenbud and Ulrich to this setting [28], and Bachmann–Wickelgren relate the work of Eisenbud–Ulrich to $\mathbb{A}^1$-enumerative problems involving excess and residual intersections [3].

3

Bézout's Theorem

In this chapter, we study the intersections of n hypersurfaces in projective n-space over an arbitrary perfect field k. Classically, Bézout's theorem addresses such intersections over an algebraically closed field.

Theorem 3.1 (Bézout's theorem)**.** *Fix an algebraically closed field k. Let $f_1, \ldots, f_n$ be hypersurfaces in $\mathbb{P}^n$, and let d_i be the degree of f_i for each i. Assume that $f_1, \ldots, f_n$ have no common components, so that $f_1 \cap \ldots \cap f_n$ is a finite set. Then, summing over the intersection points of $f_1, \ldots, f_n$, we have*

$$\sum_{points} i_p(f_1, \ldots, f_n) = d_1 \cdots d_n, \tag{3.1}$$

where $i_p(f_1, \ldots, f_n)$ is the intersection multiplicity of $f_1, \ldots, f_n$ at p.

Working over an algebraically closed field is necessary for this result.[1] Indeed, consider the intersection of a conic and a cubic shown in Figure 3.1. Over any field,

[1] Over non-algebraically closed fields, one may modify Equation 3.1 by multiplying the intersection multiplicity $i_p(f_1, \ldots, f_n)$ by the degree of the residue field $[k(p) : k]$ as described in [31, Proposition 8.4]. However, since each point p splits into $[k(p) : k]$ points in the algebraic closure of k, this simply counts the geometric intersection points as in Theorem 3.1.

these two curves do not intersect on the line at infinity. Over $\mathbb{R}$, these two curves intersect exactly twice, with intersection multiplicity one at each of the intersection points. This number falls short of the six complex intersection points, even when counted with multiplicity. The results of this chapter include a version of Bézout's theorem over $\mathbb{R}$, which will impose a relation on the gradients of these curves at their intersection points.

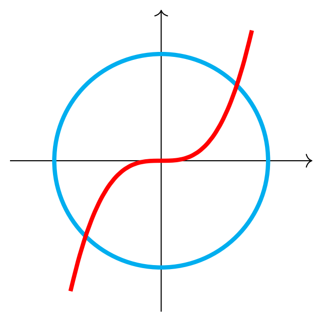

FIGURE 3.1: A conic and a cubic over $\mathbb{R}$.

Our approach to generalize Bézout's theorem follows the general philosophy of [37]. Any section σ of the vector bundle $\mathcal{O}_{d_1,\ldots,d_n} := \bigoplus_{i=1}^n \mathcal{O}(d_i) \to \mathbb{P}^n$ determines an n-tuple $(f_1, \ldots, f_n)$ of homogeneous polynomials of degree $d_1, \ldots, d_n$, respectively. The vanishing of each f_i gives a hypersurface of $\mathbb{P}^n$, which we will also denote f_i. The section σ vanishes precisely when $f_1, \ldots, f_n$ intersect, which suggests a connection to Bézout's theorem.

$\mathbb{A}^1$-homotopy theory provides a powerful tool with which to study such sections. Morel developed an $\mathbb{A}^1$-homotopy theoretic analog of the local Brouwer degree [52], which Kass and Wickelgren used to study an Euler class e of vector bundles in the context of enumerative algebraic geometry [37].[2] When $\mathcal{O}_{d_1,\ldots,d_n}$ is relatively orientable over $\mathbb{P}^n$ (that is, when $\sum_{i=1}^n d_i \equiv n+1 \mod 2$), we compute $e(\mathcal{O}_{d_1,\ldots,d_n})$, which gives an equation involving the sum of local $\mathbb{A}^1$-degrees of a generic section

[2] There are various related Euler classes in arithmetic geometry, such as those appearing in [1, 4, 29, 33, 41, 52]. See [37, Section 1.1] for a discussion.

at its points of vanishing. We also give a geometric description of the local $\mathbb{A}^1$-degree for transverse sections of $\mathcal{O}_{d_1,\ldots,d_n}$. This geometric information, paired with the equation coming from $e(\mathcal{O}_{d_1,\ldots,d_n})$, generalizes Bézout's theorem. When $\mathcal{O}_{d_1,\ldots,d_n}$ is not relatively orientable over $\mathbb{P}^n$ (that is, when $\sum_{i=1}^n d_i \not\equiv n+1 \mod 2$), we give a relative orientation of $\mathcal{O}_{d_1,\ldots,d_n}$ relative to the divisor $D = \{x_0 = 0\}$ in the sense of Larson and Vogt [40]. This allows us to compute the local degree of sections that do not vanish on D. However, we do not address the question of Euler classes in the non-relatively orientable case.

The local $\mathbb{A}^1$-degree is valued in the Grothendieck–Witt group $\mathrm{GW}(k)$ of symmetric, non-degenerate bilinear forms over k, so our enriched version of Bézout's theorem will be an equality in $\mathrm{GW}(k)$. We assume throughout this chapter that k is a perfect field, which ensures that all algebraic extensions of k are separable.

Theorem 3.2. *Let $\sum_{i=1}^n d_i \equiv n+1 \mod 2$, and let $f_1,\ldots,f_n$ be hypersurfaces of $\mathbb{P}^n$ of degree $d_1,\ldots,d_n$ that intersect transversely. Given an intersection point p of $f_1,\ldots,f_n$, let $J(p)$ be the signed volume of the parallelpiped determined by the gradient vectors of $f_1,\ldots,f_n$ at p. Then summing over the intersection points of $f_1,\ldots,f_n$, we have*

$$\sum_{\textit{points}} \mathrm{Tr}_{k(p)/k}\langle J(p)\rangle = \frac{d_1\cdots d_n}{2}\cdot\mathbb{H}, \tag{3.2}$$

where $\mathrm{Tr}_{k(p)/k} : \mathrm{GW}(k(p)) \to \mathrm{GW}(k)$ is given by post-composing with the field trace.

Taking the rank, signature, and discriminant of the Equation 3.2 gives us Bézout's theorem over $\mathbb{C}, \mathbb{R}$, and $\mathbb{F}_q$, respectively. We apply similar techniques to also study Bézout's theorem over $\mathbb{C}(\!(t)\!)$ and $\mathbb{Q}$. Kass and Wickelgren showed that Morel's local $\mathbb{A}^1$-degree is equivalent to a class of Eisenbud, Levine, and Khimshiashvili [36], which allows us to make the necessary computations without explicitly using $\mathbb{A}^1$-homotopy theory. This work fits into the growing field of $\mathbb{A}^1$-enumerative geometry, which is the enrichment of classical theorems from enumerative geometry via $\mathbb{A}^1$-homotopy theory. Related results include [2, 6, 36, 37, 40, 41, 43, 62, 65].

The layout of the chapter is as follows. In Section 3.1, we introduce notation and conventions for the chapter. In Section 3.2, we recall definitions and make computations about *relative orientability*, which is necessary for computations and proofs in Sections 3.3 and 3.4. In Section 3.3, we calculate the Euler class, and we discuss the geometric information carried by the local degree in Section 3.4. Finally, we discuss Bézout's theorem over $\mathbb{C}$, $\mathbb{R}$, finite fields of odd characteristic, $\mathbb{C}((t))$, and $\mathbb{Q}$ in Section 3.5. To illustrate the sort of obstructions that Bézout's theorem over $\mathbb{Q}$ provides, we show in Example 3.45 that if a line and a conic in $\mathbb{P}^2_{\mathbb{Q}}$ meet at two distinct points, and if the area of the parallelogram determined by the normal vectors to these curves at one of the intersection points is a non-square integer $m \neq -1$, then the area of the parallelogram at the other intersection point cannot be an integer prime to m.

3.0.1 Related work

Chen [18, Section 3] studies Bézout's theorem in $\mathbb{P}^{2n}_{\mathbb{R}}$ as a consequence of a generalized Bézout's theorem over $\mathbb{C}$ [18, Theorem 2.1]. In particular, Chen discusses that over $\mathbb{R}$, intersection multiplicities can be negative numbers [18, Remark 2.2] and shows that if $f, g \in \mathbb{R}^2$ meet transversely at p, then the $\mathbb{R}$-intersection multiplicity of f and g at p is the sign of the Jacobian of f and g [18, Proposition 3.1]. Our work in Section 3.5.2 generalizes these latter observations.

3.1 Notation and conventions

Throughout this chapter, we let k be a perfect field. We denote projective n-space over k by $\mathbb{P}^n_k = \mathrm{Proj}(k[x_0, \ldots, x_n])$. When the base field is clear from context, we may write $\mathbb{P}^n$ instead of $\mathbb{P}^n_k$. Given a rank r vector bundle E, the *determinant bundle* of E is the r-fold wedge product

$$\det E = \underbrace{E \wedge \cdots \wedge E}_{r \text{ times}}.$$

3.1.1 Standard cover

Let $U_0, \ldots, U_n$ be the standard affine open subspaces of $\mathbb{P}^n$ given by $U_i = \{[p_0 : \cdots : p_n] \in \mathbb{P}^n : p_i \neq 0\}$. Let $\varphi_0, \ldots, \varphi_n$ be the standard local coordinates of $U_0, \ldots, U_n$, where $\varphi_i : U_i \to \mathbb{A}^n$ is given by $\varphi_i([p_0 : \cdots : p_n]) = (\frac{p_0}{p_i}, \ldots, \frac{p_{i-1}}{p_i}, \frac{p_{i+1}}{p_i}, \ldots, \frac{p_n}{p_i})$. We call $\{(U_i, \varphi_i)\}$ the *standard cover* of $\mathbb{P}^n$.

3.1.2 Twisting sheaves

We denote the twisting sheaf $\mathcal{O}_{\mathbb{P}^n}(d\{x_0 = 0\})$ by $\mathcal{O}(d)$. Under this definition, we remark that $\mathcal{O}(d)$ is locally trivialized by $(\frac{x_i}{x_0})^d$ over U_i. If $d \geqslant 0$, the vector space of global sections $H^0(\mathbb{P}^n, \mathcal{O}(d))$ is isomorphic to the vector space of homogeneous polynomials in $k[x_0, \ldots, x_n]$ of degree d. Indeed, given $h \in k[x_0, \ldots, x_n]_{(d)}$, we have a global section σ of $\mathcal{O}(d)$, which is given in the local trivializations by $\sigma|_{U_i} = h/x_i^d$.

In this chapter, we will often consider global sections of $\mathcal{O}_{d_1,\ldots,d_n} := \bigoplus_{i=1}^n \mathcal{O}(d_i)$. By the above identification of $H^0(\mathbb{P}^n, \mathcal{O}(d))$ and $k[x_0, \ldots, x_n]_{(d)}$, we may thus write a section as $\sigma = (f_1, \ldots, f_n)$, where $f_i \in k[x_0, \ldots, x_n]_{(d_i)}$.

3.2 Relative orientations

Let $f_1, \ldots, f_n$ be hypersurfaces in $\mathbb{P}^n$. Bézout's theorem equates a *fixed value* with the sum (over the intersection locus of $f_1, \ldots, f_n$) of some *geometric information* about $f_1, \ldots, f_n$ at each intersection point. Classically (that is, over an algebraically closed field), the *fixed value* is the product of the degrees of each f_i, and the *geometric information* at each intersection point is the intersection multiplicity of $f_1, \ldots, f_n$. Over an arbitrary perfect field, an $\mathbb{A}^1$-homotopy theoretic Euler class will give us a particular bilinear form as our *fixed value*, and the local $\mathbb{A}^1$-degree will give us our *geometric information*. We compute the Euler class in Section 3.3, and we discuss the local degree in Section 3.4. In this section, we recall definitions and make computations that are required for Sections 3.3 and 3.4. We first start with some definitions.

Definition 3.3. [37, Definition 16] A *relative orientation* of a vector bundle V on a scheme X is a pair (L, j) of a line bundle L and an isomorphism $j : L^{\otimes 2} \to \operatorname{Hom}(\det \mathcal{T}X, \det V)$, where $\mathcal{T}X \to X$ is the tangent bundle. We say that V is *relatively orientable* if V has a relative orientation. Moreover, on an open set $U \subseteq X$, a section of $\operatorname{Hom}(\det \mathcal{T}X, \det V)$ is called a *square* if its image under $H^0(U, \operatorname{Hom}(\det \mathcal{T}X, \det V)) \cong H^0(U, L^{\otimes 2})$ is a tensor square of an element in $H^0(U, L)$.

The relative orientability of the vector bundle $\mathcal{O}_{d_1,\ldots,d_n} \to \mathbb{P}^n$ depends on $d_1, \ldots, d_n$, and n in the following way.

Proposition 3.4. *The vector bundle $\mathcal{O}_{d_1,\ldots,d_n} \to \mathbb{P}^n$ is relatively orientable if and only if $\sum_{i=1}^n d_i \equiv n+1 \mod 2$.*

Proof. Since $\det \mathcal{O}_{d_1,\ldots,d_n} \cong \bigotimes_{i=1}^n \mathcal{O}(d_i)$, we have that

$$\begin{aligned}\operatorname{Hom}(\det \mathcal{T}\mathbb{P}^n, \det \mathcal{O}_{d_1,\ldots,d_n}) &\cong \det \mathcal{O}_{d_1,\ldots,d_n} \otimes (\det \mathcal{T}\mathbb{P}^n)^\vee \\ &\cong \mathcal{O}(-n-1+\textstyle\sum_{i=1}^n d_i).\end{aligned}$$

Thus $\mathcal{O}(-n-1+\sum_{i=1}^n d_i)$ is a square if and only if $-n-1+\sum_{i=1}^n d_i$ is even, in which case $\mathcal{O}(-n-1+\sum_{i=1}^n d_i) \cong \mathcal{O}((-n-1+\sum_{i=1}^n d_i)/2)^{\otimes 2}$. □

Remark 3.5. We note that if $\sum_{i=1}^n d_i \equiv n+1 \mod 2$, then at least one of $d_1, \ldots, d_n$ must be even. Indeed, suppose all of $d_1, \ldots, d_n$ are odd. Then $d_i \equiv 1 \mod 2$, so $\sum_{i=1}^n d_i \equiv n \mod 2$.

When V is not relatively orientable, we have the following definition of Larson and Vogt.

Definition 3.6. [40, Definition 2.2] A *relative orientation relative to an effective divisor* D of a vector bundle V on a smooth projective scheme X is a pair (L, j) of a line bundle L and an isomorphism $j : L^{\otimes 2} \to \mathrm{Hom}(\det \mathcal{T}X, \det V) \otimes \mathcal{O}(D)$.

In $\mathbb{A}^1$-homotopy theory, one frequently uses the *Nisnevich topology.* For this chapter, we will only need the following definitions.

Definition 3.7. [37, Definition 17] Let X be a scheme of dimension n, and let $U \subseteq X$ be an open neighborhood of a point $p \in X$. An étale map $\varphi : U \to \mathbb{A}^n_k$ is called *Nisnevich coordinates* about p if φ induces an isomorphism between the residue field of p and the residue field of $\varphi(p)$.

Definition 3.8. [37, Definition 19] Let V be a vector bundle on a scheme X, and let $U \subseteq X$ be an open affine subset. Given Nisnevich coordinates φ on U and a relative orientation (L, j) of V, we have a distinguished basis element of $\det \mathcal{T}X|_U$. A local trivialization of $V|_U$ is called *compatible* with the Nisnevich coordinates and relative orientation if the element of $\mathrm{Hom}(\det \mathcal{T}X|_U, \det V|_U)$ taking the distinguished basis element of $\det \mathcal{T}X|_U$ to the distinguished basis element of $\det V|_U$ (determined by the specified local trivialization of $V|_U$) is a square (in the sense of Definition 3.3).

We can generalize the above definition to discuss *compatibility* in the case of a relative orientation relative to an effective Cartier divisor.

Definition 3.9. Let V be a vector bundle on a smooth projective scheme X, and let $U \subseteq X$ be an open affine subset. Given Nisnevich coordinates φ on U and a relative orientation (L, j) relative to an effective Cartier divisor D, we have a distinguished basis element of $\det \mathcal{T}X|_U$. A local trivialization of $V|_U$ is called *compatible* with the Nisnevich coordinates and relative orientation relative to D if $\alpha \otimes 1_D$ is a square, where 1_D is the canonical section of $\mathcal{O}(D)$ [63, Definition 01WX (2)] and α is the element of $\mathrm{Hom}(\det \mathcal{T}X|_U, \det V|_U)$ taking the distinguished basis element of

$\det \mathcal{T}X|_U$ to the distinguished basis element of $\det V|_U$ (determined by the specified local trivialization of $V|_U$).

We will show that a twist of the standard cover $\{(U_i, \varphi_i)\}$ of $\mathbb{P}^n$ (see Section 3.1.1) gives Nisnevich coordinates. This twist will be denoted $\{(U_i, \tilde{\varphi}_i)\}$, with $\tilde{\varphi}_0 = \varphi_0$ and

$$\tilde{\varphi}_i([p_0 : \cdots : p_n]) = ((-1)^i \tfrac{p_0}{p_i}, \ldots, \tfrac{p_{i-1}}{p_i}, \tfrac{p_{i+1}}{p_i}, \ldots, \tfrac{p_n}{p_i}).$$

The reason for working with these twisted coordinates instead of the standard coordinates is to ensure compatibility with the local trivializations of $\mathcal{O}_{d_1,\ldots,d_n}$, as shown in Lemmas 3.12 and 3.13. We will also describe the distinguished basis elements of $\det \mathcal{T}\mathbb{P}^n|_{U_i}$ and $\det \mathcal{O}_{d_1,\ldots,d_n}|_{U_i}$ coming from $\tilde{\varphi}_i$ and the local trivialization given in Section 3.1.2, respectively.

Proposition 3.10. *The twisted covering maps $\tilde{\varphi}_i : U_i \to \mathbb{A}^n_k$ are Nisnevich coordinates. Moreover, φ_i determines the distinguished basis element $(-1)^i \cdot \partial_i := (-1)^i \bigwedge_{j\neq i} \frac{\partial}{\partial(x_j/x_i)}$ of $\det \mathcal{T}\mathbb{P}^n|_{U_i}$ with transition functions $\det g_{ij} := (-1)^{i+j}(\frac{x_i}{x_j})^{n+1}$.*

Proof. By construction, $\tilde{\varphi}_i : U_i \to \mathbb{A}^n_k$ is an isomorphism, so $\tilde{\varphi}_i$ is étale and induces an isomorphism $k(p) \cong k(\varphi(p))$ for all $p \in U_i$. Recall that $\mathcal{T}\mathbb{A}^n$ has the standard trivializations $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\}$. Since $\tilde{\varphi}_i$ induces an isomorphism $\mathcal{T}\mathbb{P}^n|_{U_i} \cong \mathcal{T}\mathbb{A}^n$, we may pull back the standard trivializations $\mathcal{T}\mathbb{A}^n \to \mathbb{A}^n$ by $\tilde{\varphi}_i$ to obtain the twisted trivializations $\{(-1)^i \partial_{0/i}, \ldots, \partial_{(i-1)/i}, \partial_{(i+1)/i}, \ldots, \partial_{n/i}\}$, where $\partial_{j/i} = \frac{\partial}{\partial(x_j/x_i)}$. It follows that $\det \mathcal{T}\mathbb{P}^n|_{U_i}$ is trivialized by $(-1)^i \bigwedge_{j\neq i} \partial_{j/i}$. Finally, we consider the transition functions $\det g_{ij} : \det \mathcal{T}\mathbb{P}^n|_{U_j} \to \det \mathcal{T}\mathbb{P}^n|_{U_i}$. These transition functions will come from the transition functions $g_{ij} : \mathcal{T}\mathbb{P}^n|_{U_j} \to \mathcal{T}\mathbb{P}^n|_{U_i}$. A few calculus computations show us that, for $k \neq i, j$, we have

$$\partial_{k/i} = \tfrac{x_i}{x_j} \cdot \partial_{k/j},$$

$$\partial_{j/i} = -(\tfrac{x_i}{x_j})^2 \cdot \partial_{i/j} - \sum_{k\neq i,j} \tfrac{x_i x_k}{x_j^2} \cdot \partial_{k/j}.$$

Thus for fixed i, j, we have $\bigwedge_{k\neq i} \partial_{k/i} = -(\frac{x_i}{x_j})^{n+1} \bigwedge_{k\neq j} \partial_{k/j}$. The trivializations $(-1)^i \cdot \partial_i := (-1)^i \bigwedge_{j\neq i} \partial_{j/i}$ of $\det \mathcal{T}\mathbb{P}^n|_{U_i}$ are local trivializations of $\det \mathcal{T}\mathbb{P}^n$ compatible with the transition functions $\det g_{ij} = (-1)^{i+j}(\frac{x_i}{x_j})^{n+1}$. In other words, $\det \mathcal{T}\mathbb{P}^n|_{U_i}$ is one-dimensional with $(-1)^i \cdot \partial_i$ as its distinguished basis element. □

Proposition 3.11. *The local trivialization $(\frac{x_i}{x_0})^{d_1} \oplus \cdots \oplus (\frac{x_i}{x_0})^{d_n}$ of $\mathcal{O}_{d_1,\ldots,d_n}|_{U_i}$ determines the distinguished basis element $(\frac{x_i}{x_0})^{d_1+\ldots+d_n}$ of $\det \mathcal{O}_{d_1,\ldots,d_n}|_{U_i}$ with transition functions $\det h_{ij} := (\frac{x_i}{x_j})^{d_1+\ldots+d_n}$.*

Proof. Since $\mathcal{O}(d)|_{U_i}$ is trivialized by $(\frac{x_i}{x_0})^d$, the vector bundle $\mathcal{O}_{d_1,\ldots,d_n}|_{U_i}$ is trivialized by $(\frac{x_i}{x_0})^{d_1} \oplus \cdots \oplus (\frac{x_i}{x_0})^{d_n}$. The transition functions $h_{ij} : \mathcal{O}(d)|_{U_j} \to \mathcal{O}(d)|_{U_i}$ are given by $(\frac{x_i}{x_j})^d$, so the transition functions $\oplus h_{ij} : \mathcal{O}_{d_1,\ldots,d_n}|_{U_j} \to \mathcal{O}_{d_1,\ldots,d_n}|_{U_i}$ are given by $(\frac{x_i}{x_j})^{d_1} \oplus \cdots \oplus (\frac{x_i}{x_j})^{d_n}$. Finally, recall that $\det \mathcal{O}_{d_1,\ldots,d_n} \cong \mathcal{O}(d_1) \otimes \cdots \otimes \mathcal{O}(d_n) \cong \mathcal{O}(d_1 + \ldots + d_n)$. Thus $\det \mathcal{O}_{d_1,\ldots,d_n}|_{U_i}$ is trivialized by $(\frac{x_i}{x_0})^{d_1} \otimes \cdots \otimes (\frac{x_i}{x_0})^{d_n} \cong (\frac{x_i}{x_0})^{d_1+\ldots+d_n}$, and the transition functions $\det h_{ij} : \det \mathcal{O}_{d_1,\ldots,d_n}|_{U_j} \to \det \mathcal{O}_{d_1,\ldots,d_n}|_{U_i}$ are given by $(\frac{x_i}{x_j})^{d_1} \otimes \cdots \otimes (\frac{x_i}{x_j})^{d_n} \cong (\frac{x_i}{x_j})^{d_1+\ldots+d_n}$. □

3.2.1 Relatively orientable case

Let $N = -n - 1 + \sum_{i=1}^n d_i$, and assume $N \equiv 0 \mod 2$, so that $\mathcal{O}_{d_1,\ldots,d_n} \to \mathbb{P}^n$ is relatively orientable by Proposition 3.4. We will give a relative orientation of $\mathcal{O}_{d_1,\ldots,d_n}$ and show that the local trivializations of $\mathcal{O}_{d_1,\ldots,d_n}$ discussed in Proposition 3.11 are compatible with this relative orientation and the Nisnevich coordinates coming from our twisted cover $\{(U_i, \tilde{\varphi}_i)\}$.

For our relative orientation, we give an isomorphism

$$\psi : \mathcal{O}(N/2)^{\otimes 2} \to \mathrm{Hom}(\det \mathcal{T}\mathbb{P}^n, \det \mathcal{O}_{d_1,\ldots,d_n})$$

by defining $\psi|_{U_i}$ for each i. Since $\mathcal{O}(N/2)^{\otimes 2}|_{U_i}$ is generated by $(\frac{x_i}{x_0})^{N/2} \otimes (\frac{x_i}{x_0})^{N/2}$, it suffices to define $\alpha_i := \psi|_{U_i}((\frac{x_i}{x_0})^{N/2} \otimes (\frac{x_i}{x_0})^{N/2})$, which is a homomorphism from

$\det \mathcal{T}\mathbb{P}^n|_{U_i}$ to $\det \mathcal{O}_{d_1,\ldots,d_n}|_{U_i}$. These are both one-dimensional as shown in Propositions 3.10 and 3.11, so we may define α_i to be the homomorphism taking $(-1)^i \cdot \partial_i$ to $(\frac{x_i}{x_0})^{d_1+\ldots+d_n}$. To show that ψ is well-defined, we need to show that on $U_i \cap U_j$, the maps $\psi|_{U_i}$ and $\psi|_{U_j}$ differ by the transition function $(\frac{x_i}{x_j})^{N/2} \otimes (\frac{x_i}{x_j})^{N/2} : \mathcal{O}(N/2)^{\otimes 2}|_{U_j} \to \mathcal{O}(N/2)^{\otimes 2}|_{U_i}$. In other words, we need to show that $\alpha_i = (\frac{x_i}{x_j})^N \alpha_j$ on $U_i \cap U_j$. To this end, let $\det g_{ij}$ and $\det h_{ij}$ be the transition functions given in Propositions 3.10 and 3.11 and note that

$$\begin{aligned}
\alpha_i \circ \det g_{ij}((-1)^j \cdot \partial_j) &= \alpha_i((-1)^i (\tfrac{x_j}{x_i})^{n+1} \cdot \partial_i) \\
&= (\tfrac{x_j}{x_i})^{n+1} (\tfrac{x_i}{x_0})^{d_1+\ldots+d_n} \\
&= (\tfrac{x_j}{x_i})^{n+1} \det h_{ij} (\tfrac{x_j}{x_0})^{d_1+\ldots+d_n} \\
&= (\tfrac{x_i}{x_j})^{-n-1} (\tfrac{x_i}{x_j})^{d_1+\ldots+d_n} (\tfrac{x_j}{x_0})^{d_1+\ldots+d_n} \\
&= (\tfrac{x_i}{x_j})^N \alpha_j((-1)^j \cdot \partial_j).
\end{aligned}$$

Thus $\alpha_i = (\frac{x_i}{x_j})^N \alpha_j$, as desired. In fact, we have proved the following lemma.

Lemma 3.12. *The local trivializations* $(\frac{x_i}{x_0})^{d_1} \oplus \cdots \oplus (\frac{x_i}{x_0})^{d_n}$ *of* $\mathcal{O}_{d_1,\ldots,d_n}|_{U_i}$ *are compatible with the Nisnevich coordinates* $\{(U_i, \tilde{\varphi}_i)\}$ *and the relative orientation* $(\mathcal{O}(N/2), \psi)$ *of* $\mathcal{O}_{d_1,\ldots,d_n} \to \mathbb{P}^n$.

Proof. By construction, α_i is the element of $\mathrm{Hom}(\det \mathcal{T}\mathbb{P}^n|_{U_i}, \det \mathcal{O}_{d_1,\ldots,d_n}|_{U_i})$ taking the distinguished basis element $(-1)^i \cdot \partial_i$ of $\det \mathcal{T}\mathbb{P}^n|_{U_i}$ to the distinguished basis element $(\frac{x_i}{x_0})^{d_1+\ldots+d_n}$ of $\det \mathcal{O}_{d_1,\ldots,d_n}|_{U_i}$. The relative orientation $(\mathcal{O}(N/2), \psi)$ was built such that $\psi|_{U_i}((\frac{x_i}{x_0})^{N/2} \otimes (\frac{x_i}{x_0})^{N/2}) = \alpha_i$, so α_i is a tensor square in $\mathcal{O}(N/2)^{\otimes 2}|_{U_i}$. □

3.2.2 Non-relatively orientable case

Let $N = -n - 1 + \sum_{i=1}^n d_i$, and assume $N \not\equiv 0 \mod 2$. In this case, $\mathcal{O}_{d_1,\ldots,d_n} \to \mathbb{P}^n$ is not relatively orientable, since there is no line bundle of the form $\mathcal{O}(N/2)$ when $N/2$

is not an integer. However, we will show that $\mathcal{O}_{d_1,\ldots,d_n} \to \mathbb{P}^n$ is relatively orientable relative to the effective Cartier divisor $D = \{x_0 = 0\}$ of $\mathbb{P}^n$. Figuratively, this divisor gives us a geometric horizon relative to which we can orient our hypersurfaces in projective space.

We have chosen the divisor $D = \{x_0 = 0\}$ so that the local trivializations of $\mathcal{O}(D)$ work nicely with our other twisting sheaves. In particular, we have

$$\mathrm{Hom}(\det \mathcal{T}\mathbb{P}^n, \det \mathcal{O}_{d_1,\ldots,d_n}) \otimes \mathcal{O}(D) \cong \mathcal{O}(N+1).$$

Since $N+1 \equiv 0 \mod 2$, the bundle $\mathrm{Hom}(\det \mathcal{T}\mathbb{P}^n, \det \mathcal{O}_{d_1,\ldots,d_n}) \otimes \mathcal{O}(D)$ is the tensor square of the line bundle $\mathcal{O}(\frac{N+1}{2})$. We may thus apply the work of Section 3.2.1 to get a relative orientation $(\mathcal{O}(\frac{N+1}{2}), \tilde{\psi})$ of $\mathcal{O}_{d_1,\ldots,d_n}$ relative to the divisor D, as well as local trivializations of $\mathcal{O}_{d_1,\ldots,d_n}$ compatible with our Nisnevich coordinates $\{(U_i, \tilde{\varphi}_i)\}$ and our relative orientation $(\mathcal{O}(\frac{N+1}{2}), \tilde{\psi})$.

Lemma 3.13. *The local trivialization* $(\frac{x_i}{x_0})^{d_1} \oplus \cdots \oplus (\frac{x_i}{x_0})^{d_n}$ *of* $\mathcal{O}_{d_1,\ldots,d_n}|_{U_i}$ *are compatible with the Nisnevich coordinates* $\{(U_i, \tilde{\varphi}_i)\}$ *and the relative orientation* $(\mathcal{O}(\frac{N+1}{2}), \tilde{\psi})$, *where* $\tilde{\psi}$ *is given locally by* $\tilde{\psi}|_{U_i}((\frac{x_i}{x_0})^{(N+1)/2} \otimes (\frac{x_i}{x_0})^{(N+1)/2}) = \alpha_i \otimes \frac{x_i}{x_0}$.

Proof. The canonical section 1_D of $\mathcal{O}(D)$ is locally given by $\frac{x_i}{x_0}$. By construction, $\alpha_i((-1)^i \cdot \partial_i) \otimes \frac{x_i}{x_0} = (\frac{x_i}{x_0})^{d_1+\ldots+d_n+1}$, so we have $\alpha_i \otimes \frac{x_i}{x_0} = (\frac{x_i}{x_j})^{N+1} \alpha_j \otimes \frac{x_j}{x_0}$ on $U_i \cap U_j$ by Lemma 3.12. Thus the maps $\tilde{\psi}|_{U_i}$ and $\tilde{\psi}|_{U_j}$ differ by the transition function $(\frac{x_i}{x_j})^{(N+1)/2} \otimes (\frac{x_i}{x_j})^{(N+1)/2} : \mathcal{O}(\frac{N+1}{2})^{\otimes 2}|_{U_j} \to \mathcal{O}(\frac{N+1}{2})^{\otimes 2}|_{U_i}$, so the relative orientation $(\mathcal{O}(\frac{N+1}{2}), \tilde{\psi})$ relative to the divisor D is well-defined. This relative orientation was constructed such that $\alpha_i \otimes \frac{x_i}{x_0}$ is a square. $\square$

3.3 Euler class

In Section 3.2, we gave a relative orientation (possibly relative to an effective Cartier divisor) of $\mathcal{O}_{d_1,\ldots,d_n} \to \mathbb{P}^n$, as well as local trivializations of this bundle compatible with the given relative orientation and our twisted Nisnevich coordinates. This data allows us to compute the local degree of sections of $\mathcal{O}_{d_1,\ldots,d_n}$. We can also define an Euler class of this vector bundle, paired with a given section, by computing the sum of local degrees of the section. When this sum does not depend on our choice of section, the Euler class gives us an invariant associated to the vector bundle at hand. This invariant will correspond to the enumerative *fixed value* discussed at the beginning of Section 3.2.

We first discuss how to compute the local degree of a section. Let V be a vector bundle on a scheme X of dimension n, and suppose that we have Nisnevich coordinates, a relative orientation (possibly relative to an effective Cartier divisor) of V, and compatible local trivializations of V. If σ is a section of V with isolated zero $p \in X$, then take an open affine $U \subseteq X$ containing p. Under the compatible local trivialization of $V|_U$, the section $\sigma|_U$ becomes an n-tuple of functions $(f_1, \ldots, f_n) : U \to \mathbb{A}^n_k$. If $\varphi : U \to \mathbb{A}^n_k$ are the aforementioned Nisnevich coordinates, and if $\varphi|_U$ is an isomorphism, then $(f_1, \ldots, f_n) \circ \varphi^{-1}$ is an endomorphism of $\mathbb{A}^n_k$. (In general, $\varphi|_U$ may not be an isomorphism, in which case we cannot write the section σ as a polynomial map. We can, however, write our section as a polynomial map with a negligible error term. See [37, Lemmas 24–28] for details.)

We may thus compute the local degree of this endomorphism as outlined in [36, Table 1]. Kass and Wickelgren [37, Corollary 29] also show that given a relative orientation of V, the Euler class e does not depend on the choice of Nisnevich coordinates on X with compatible trivialization of V. This allows us to define $\deg_p \sigma = \deg_{\varphi(p)}(f_1, \ldots, f_n) \circ \varphi^{-1}$. Note that we must choose our neighborhood U sufficiently small, so that $\varphi^{-1}(\varphi(p)) = \{p\}$.

Definition 3.14. [37, Definition 33] Given a relatively oriented (relative to an effective Cartier divisor D) vector bundle $V \to X$ and a section σ with isolated zero locus (such that σ does not vanish on D), define the *Euler number* of (V, σ) to be

$$e(V, \sigma) = \sum_{p \in \sigma^{-1}(0)} \deg_p \sigma.$$

When $e(V, \sigma)$ does not depend on our choice of section σ, we will simply denote this by $e(V)$. It will turn out that $e(\mathcal{O}_{d_1,\ldots,d_n})$ does not depend on our choice of section when $\mathcal{O}_{d_1,\ldots,d_n} \to \mathbb{P}^n$ is relatively orientable. Let $N = -n - 1 + \sum_{i=1}^{n} d_i$, and assume $N \equiv 0 \mod 2$ so that $\mathcal{O}_{d_1,\ldots,d_n} \to \mathbb{P}^n$ is relatively orientable. We will show that $e(\mathcal{O}_{d_1,\ldots,d_n}, \sigma)$ does not depend on our choice of section. First, we need the following proposition.

Proposition 3.15. *Let* $\operatorname{char} k = 0$*. Let* $n \geqslant 1$*. If* $(f_1, \ldots, f_{i-1})$ *is a regular sequence in* $k[x_0, \ldots, x_n]$ *for* $1 \leqslant i \leqslant n$*, then*

$$Z_i := \{f_i \in H^0(\mathbb{P}^n_k, \mathcal{O}(d_i)) : (f_1, \ldots, f_{i-1}, f_i) \text{ is not a regular sequence}\}$$

has k*-codimension at least 2 in* $H^0(\mathbb{P}^n_k, \mathcal{O}(d_i))$*.*

Proof. By definition of regular sequences, f_i is not a zero divisor in $\frac{k[x_0,\ldots,x_n]}{(f_1,\ldots,f_{i-1})}$ if and only if $(f_1, \ldots, f_{i-1}, f_i)$ is a regular sequence in $k[x_0, \ldots, x_n]$. The set of zero divisors in $\frac{k[x_0,\ldots,x_n]}{(f_1,\ldots,f_{i-1})}$ is given by the union of the minimal prime ideals associated to the ideal $I := (f_1, \ldots, f_{i-1})$. Moreover, since $k[x_0, \ldots, x_n]$ is Noetherian, there are finitely many minimal prime ideals associated to I. Given a minimal prime $\mathfrak{p}$ associated to I, let $\mathfrak{p}_{d_i}$ denote the degree d_i part of $\mathfrak{p}$, considered as a k-vector space. If

$$\operatorname{codim}_k \mathfrak{p}_{d_i} := \dim_k H^0(\mathbb{P}^n_k, \mathcal{O}(d_i)) - \dim_k \mathfrak{p}_{d_i} \geqslant 2$$

for any minimal prime $\mathfrak{p}$ associated to I, then Z_i is a finite union of spaces of codimension at least 2. It will then follow that Z_i has codimension at least 2 in $H^0(\mathbb{P}^n_k, \mathcal{O}(d_i))$.

Let $\mathfrak{p}$ be a minimal prime ideal associated to I. Krull's height theorem implies that $\mathfrak{p}$ has height at most $i-1$, so $\mathfrak{p}$ contains at most $i-1$ linear forms that are linearly independent over k. Suppose that $\operatorname{codim}_k \mathfrak{p}_{d_i} < 2$. Then $\dim_k \mathfrak{p}_{d_i} \geqslant \binom{n+d_i}{d_i} - 1$. If $\dim_k \mathfrak{p}_{d_i} = \binom{n+d_i}{d_i}$, then $\mathfrak{p}_{d_i} = H^0(\mathbb{P}^n_k, \mathcal{O}(d_i))$ and hence $x_0^{d_i}, \ldots, x_n^{d_i} \in \mathfrak{p}_{d_i}$. Since $\mathfrak{p}$ is a prime ideal, it follows that $x_0, \ldots, x_n \in \mathfrak{p}$, so $\mathfrak{p}$ contains $n+1 > i-1$ linear forms that are linearly independent.

We may thus assume that $\dim_k \mathfrak{p}_{d_i} = \binom{n+d_i}{d_i} - 1$. Let $N = \binom{n+d_i}{d_i}$, and consider the Veronese embedding $v_{d_i} : \mathbb{P}^n_k \to \mathbb{P}^{N-1}_k$, where $\mathbb{P}^n_k \cong \mathbb{P}(H^0(\mathbb{P}^n_k, \mathcal{O}(1)))$ and $\mathbb{P}^{N-1}_k \cong \mathbb{P}(H^0(\mathbb{P}^n_k, \mathcal{O}(d_i)))$. Under the Veronese embedding, the image of $\ell \in \mathbb{P}(H^0(\mathbb{P}^n_k, \mathcal{O}(1)))$ is $v_{d_i}(\ell) = \ell^{d_i} \in \mathbb{P}(H^0(\mathbb{P}^n_k, \mathcal{O}(d_i)))$. Since $\mathfrak{p}_{d_i}$ is a codimension 1 subspace of $H^0(\mathbb{P}^n_k, \mathcal{O}(d_i))$ by assumption, we have an isomorphism $\mathbb{P}(\mathfrak{p}_{d_i}) \cong H$ for some hyperplane $H \subset \mathbb{P}^{N-1}_k$. The image $v_{d_i}(\mathbb{P}^n_k)$ of the Veronese embedding is not contained in any hyperplane, so the hyperplane section $v_{d_i}(\mathbb{P}^n_k) \cap H$ has dimension $\dim v_{d_i}(\mathbb{P}^n_k) - 1$. Since the Veronese embedding is an isomorphism onto its image, it follows that $v_{d_i}(\mathbb{P}^n_k) \cap H = v_{d_i}(X)$ for some $X \subset \mathbb{P}^n_k$ of dimension $n-1$. This allows us to pick general points $p_1, \ldots, p_n \in X$ such that $\{p_1, \ldots, p_n\}$ is not contained in any $(n-2)$-plane. Thus if $\ell_j \in H^0(\mathbb{P}^n_k, \mathcal{O}(1))$ is any lift of p_j under

$$H^0(\mathbb{P}^n_k, \mathcal{O}(1)) \to \mathbb{P}(H^0(\mathbb{P}^n_k, \mathcal{O}(1))) \cong \mathbb{P}^n_k,$$

then $\ell_1, \ldots, \ell_n$ are linearly independent over k. Moreover, since $v_{d_i}(p_j) \in H$, we have that $\ell_j^{d_i} \in \mathfrak{p}_{d_i}$. Since $\mathfrak{p}$ is a prime ideal, it follows that $\ell_1, \ldots, \ell_n \in \mathfrak{p}$, so $\mathfrak{p}$ contains $n > i-1$ linear forms that are linearly independent. By contradiction, we conclude that $\operatorname{codim}_k \mathfrak{p}_{d_i} \geqslant 2$. $\square$

We can now prove that $e(\mathcal{O}_{d_1,\ldots,d_n}, \sigma)$ does not depend on our choice of section.

Lemma 3.16. *The Euler number $e(\mathcal{O}_{d_1,\ldots,d_n}, \sigma)$ is independent of the choice of section σ.*

Proof. This follows from [2, Theorem 1.1]. However, we will also give a more direct proof of this lemma assuming $\operatorname{char} k = 0$. Given a section $\sigma \in H^0(\mathbb{P}^n_k, \mathcal{O}_{d_1,\ldots,d_n})$, let $Z(\sigma) = \{p \in \mathbb{P}^n : \sigma(p) = 0\}$ be its zero locus. We will show that $\{\sigma : Z(\sigma) \text{ is not isolated}\}$ has k-codimension at least 2 in $H^0(\mathbb{P}^n_k, \mathcal{O}_{d_1,\ldots,d_n})$. This will show that

$$H^0(\mathbb{P}^n_k, \mathcal{O}_{d_1,\ldots,d_n})\backslash\{\sigma : Z(\sigma) \text{ is not isolated}\}$$

is *connected by sections* in the sense of [37, Definition 37]. As a result, [37, Corollary 38] will imply that $e(\mathcal{O}_{d_1,\ldots,d_n}, \sigma)$ is independent of σ.

The zero locus $Z(\sigma)$ is isolated if and only if $(f_1, \ldots, f_n)$ is a regular sequence, so $Z(\sigma)$ is not isolated if and only if f_i is a zero divisor in $\frac{k[x_0,\ldots,x_n]}{(f_1,\ldots,f_{i-1})}$ for some i. Note that $H^0(\mathbb{P}^n_k, \mathcal{O}_{d_1,\ldots,d_n}) = \bigoplus_{i=1}^n H^0(\mathbb{P}^n_k, \mathcal{O}(d_i))$. Let $Z_i \subseteq H^0(\mathbb{P}^n_k, \mathcal{O}_{d_1,\ldots,d_i})$ be the set of all sections $(f_1, \ldots, f_i)$ such that $(f_1, \ldots, f_{i-1})$ is a regular sequence and f_i is a zero divisor in $\frac{k[x_0,\ldots,x_n]}{(f_1,\ldots,f_{i-1})}$. Then the set of all non-regular sequences $(f_1, \ldots, f_n)$ is given by

$$\bigcup_{i=1}^n (Z_i \oplus H^0(\mathbb{P}^n_k, \mathcal{O}_{d_{i+1},\ldots,d_n})).$$

Proposition 3.15 implies that Z_i has codimension at least 2 in $H^0(\mathbb{P}^n_k, \mathcal{O}_{d_1,\ldots,d_i})$. It follows that $Z_i \oplus H^0(\mathbb{P}^n_k, \mathcal{O}(d_{i+1}))$ has codimension at least 2 in $H^0(\mathbb{P}^n_k, \mathcal{O}_{d_1,\ldots,d_{i+1}})$. Iterating this process, we have that $Z_i \oplus H^0(\mathbb{P}^n_k, \mathcal{O}_{d_{i+1},\ldots,d_n})$ has codimension at least 2 in $H^0(\mathbb{P}^n_k, \mathcal{O}_{d_1,\ldots,d_n})$ for $i = 1, \ldots, n$. The set of all non-regular sequences is thus a finite union of sets of codimension at least 2, so $\{\sigma : Z(\sigma) \text{ is not isolated}\}$ has codimension at least 2. $\square$

We may now compute $e(\mathcal{O}_{d_1,\ldots,d_n})$.

Theorem 3.17. *We have* $e(\mathcal{O}_{d_1,\ldots,d_n}) = \frac{d_1 \cdots d_n}{2} \cdot \mathbb{H}$.

Proof. By Lemma 3.16, $e(\mathcal{O}_{d_1,\ldots,d_n})$ is independent of choice of section. Let $\sigma = (x_1^{d_1}, \ldots, x_n^{d_n})$. The zero locus of σ consists only of the point $p = [1 : 0 : \cdots : 0]$, so $e(\mathcal{O}_{d_1,\ldots,d_n}) = \deg_p \sigma$. Since $p \in U_0$, our twisted cover and local trivialization of

$\mathcal{O}_{d_1,\ldots,d_n}$ on U_0 tell us that $\deg_p \sigma = \deg_{(0,\ldots,0)}((\frac{x_1}{x_0})^{d_1}, \ldots, (\frac{x_n}{x_0})^{d_n})$. We may rewrite this local degree as a product of local degrees, yielding $\deg_p \sigma = \prod_{i=1}^n \deg_0 x^{d_i}$. By Remark 3.5, at least one of $d_1, \ldots, d_n$ is even. Since

$$\deg_0 ax^d = \begin{cases} \frac{d-1}{2} \cdot \mathbb{H} + \langle a \rangle & d \text{ odd}, \\ \frac{d}{2} \cdot \mathbb{H} & d \text{ even}, \end{cases}$$

we have that

$$\begin{aligned} \deg_p \sigma &= \prod_{d_i \text{ even}} \left(\tfrac{d_i}{2} \cdot \mathbb{H}\right) \cdot \prod_{d_i \text{ odd}} \left(\tfrac{d_i-1}{2} \cdot \mathbb{H} + \langle 1 \rangle\right) \\ &= \left(\frac{\prod_{\text{even}} d_i}{2} \cdot \mathbb{H}\right)\left(\frac{(\prod_{\text{odd}} d_i) - 1}{2} \cdot \mathbb{H} + \langle 1 \rangle\right) \\ &= \frac{d_1 \cdots d_n}{2} \cdot \mathbb{H}. \end{aligned}$$

Alternately, one can note that $\deg_p \sigma$ will be of the form $a\langle 1 \rangle + b\langle -1 \rangle$ for some $a, b \in \mathbb{Z}$. The values of a and b can then be determined by separately considering the rank and signature of $\deg_0 x^{d_i}$ for $1 \leqslant i \leqslant n$. Either approach gives us $e(\mathcal{O}_{d_1,\ldots,d_n}) = \frac{d_1 \cdots d_n}{2} \cdot \mathbb{H}$. $\square$

Remark 3.18. Theorem 3.17 also follows from [41, Theorem 7.1] as described in [62, Proposition 19].

In Section 3.4, we give a geometric interpretation of $\deg_p(f_1, \ldots, f_n)$. Paired with Theorem 3.17, this will prove Theorem 3.2.

3.4 Formulas and geometric interpretations for the local degree

In Section 3.3, we computed the Euler class of the vector bundle $\mathcal{O}_{d_1,\ldots,d_n} \to \mathbb{P}^n$. Roughly speaking, this Euler class equals the sum of local degrees over the intersection locus of $f_1, \ldots, f_n$. Once we provide a geometric interpretation of the local degree $\deg_p(f_1, \ldots, f_n)$, we will have an equation that counts geometric information concerning the intersection points of $f_1, \ldots, f_n$. This enumerative geometric equation will constitute our enriched version of Bézout's theorem.

3.4.1 Intersection multiplicity is the rank of the local degree

We set out to prove that the intersection multiplicity $i_p(f_1, \ldots, f_n)$ of $f_1, \ldots, f_n$ at p is the rank of the local degree $\deg_p(f_1, \ldots, f_n)$. We begin with a brief discussion of intersection multiplicity.

Definition 3.19. [61, Definition 4.1] Given $p \in \mathbb{P}^n$ and hypersurfaces $f_1, \ldots, f_n$, the *intersection multiplicity* of $f_1, \ldots, f_n$ at p is given by

$$i_p(f_1, \ldots, f_n) = \dim_k \mathcal{O}_{\mathbb{P}^n,p}/(f_1, \ldots, f_n).$$

We next prove that this intersection multiplicity agrees with the rank of the local degree $\deg_p(f_1, \ldots, f_n)$. This will follow from the fact that both the intersection multiplicity and the local degree are given by local computations.

Proposition 3.20. *Let $f_1, \ldots, f_n$ be hypersurfaces in $\mathbb{P}^n$ that are all non-singular at a common intersection point p. Moreover, assume that $f_1, \ldots, f_n$ do not share a common component, so that $f_1 \cap \cdots \cap f_n$ is a finite set. Then*

$$\operatorname{rank} \deg_p(f_1, \ldots, f_n) = i_p(f_1, \ldots, f_n).$$

Proof. Let U be an affine neighborhood of p with local coordinates $\varphi : U \to \mathbb{A}^n$. An algorithmic method for computing $\deg_p(f_1, \ldots, f_n)$ is outlined in [36, Table 1]. In this method, we have that $\operatorname{rank} \deg_p(f_1, \ldots, f_n) = \dim_k \varphi_*(\mathcal{O}_{U,p}/(f_1, \ldots, f_n))$. (In the notation of [37], $\varphi_*(\mathcal{O}_{U,p}/(f_1, \ldots, f_n))$ corresponds to Q_p. The assumption that $f_1 \cap \ldots \cap f_n$ be a finite set ensures that p is an isolated zero, so that $\dim_k Q_p$ is finite.) First, we have that $\mathcal{O}_{\mathbb{P}^n,p}/(f_1, \ldots, f_n)$ is isomorphic (as a k-algebra) to $\mathcal{O}_{U,p}/(f_1, \ldots, f_n)$. Since $\varphi : U \to \mathbb{A}^n$ is an isomorphism, we also have that $\mathcal{O}_{U,p}/(f_1, \ldots, f_n)$ and $\varphi_*(\mathcal{O}_{U,p}/(f_1, \ldots, f_n))$ are isomorphic as k-algebras. Thus $\operatorname{rank} \deg_p(f_1, \ldots, f_n) = i_p(f_1, \ldots, f_n)$, as desired. $\square$

3.4.2 Transverse intersections

When the hypersurfaces $f_1, \ldots, f_n$ intersect transversely at a point p, the local degree $\deg_p(f_1, \ldots, f_n)$ has a geometric interpretation in terms of the gradient directions of each f_i at p. Intuitively, $f_1, \ldots, f_n$ intersect transversely at p as subschemes of $\mathbb{P}^n$ if their tangent spaces at p overlap as little as possible. This idea can be made rigorous by the following definition.

Definition 3.21. [25, p. 18] The subschemes $f_1, \ldots, f_n$ of $\mathbb{P}^n$ *intersect transversely at* p if each f_i is smooth at p and if $\operatorname{codim}(\bigcap_i T_p f_i) = \sum_i \operatorname{codim} T_p f_i$, where $\operatorname{codim} T_p f_i$ refers to the codimension of $T_p f_i$ as a subspace of the vector space $T_p \mathbb{P}^n_k \cong k(p)^n$.

Transverse intersections of n hypersurfaces in $\mathbb{P}^n$ are completely characterized by their intersection multiplicity.

Proposition 3.22. *The hypersurfaces $f_1, \ldots, f_n$ of $\mathbb{P}^n_k$ intersect transversely at a point p if and only if $i_p(f_1, \ldots, f_n) = [k(p) : k]$. In particular, if p is a k-rational point, then $f_1, \ldots, f_n$ intersect transversely at p if and only if $i_p(f_1, \ldots, f_n) = 1$.*

Proof. Let $\mathfrak{m}_p$ be the maximal ideal of $k[x_0, \ldots, x_n]$ corresponding to the point p. Since the intersection multiplicity $i_p(f_1, \ldots, f_n)$ is locally defined, we work in the local ring $k[x_0, \ldots, x_n]_{\mathfrak{m}_p} = \mathcal{O}_{\mathbb{P}^n,p}$. The polynomials $f_1, \ldots, f_n$ are *local parameters* at p in the sense of [61, Section 2.1] if and only if $(f_1, \ldots, f_n) = \mathfrak{m}_p$ (see [61, Theorem 2.5]). By [61, Theorem 2.4], we have $(f_1, \ldots, f_n) = \mathfrak{m}_p$ if and only if $f_1, \ldots, f_n$ intersect transversely at p. Thus if $f_1, \ldots, f_n$ intersect transversely at p, then $(f_1, \ldots, f_n) = \mathfrak{m}_p$ and hence $i_p(f_1, \ldots, f_n) = \dim_k \mathcal{O}_{\mathbb{P}^n,p}/\mathfrak{m}_p = [k(p) : k]$. On the other hand, the fact that $f_1, \ldots, f_n$ vanish at p implies that $(f_1, \ldots, f_n) \subseteq \mathfrak{m}_p$. Consequently, if $i_p(f_1, \ldots, f_n) = [k(p) : k] = \dim_k \mathcal{O}_{\mathbb{P}^n,p}/\mathfrak{m}_p$, then we have $(f_1, \ldots, f_n) = \mathfrak{m}_p$ and hence $f_1, \ldots, f_n$ intersect transversely at p. $\square$

By Proposition 3.20, it follows that $\operatorname{rank} \deg_p(f_1, \ldots, f_n) = [k(p) : k]$ at transverse intersection points. As mentioned previously, $f_1, \ldots, f_n$ intersect transversely at p if and only if $(f_1, \ldots, f_n) = \mathfrak{m}_p$, which is equivalent to p being a simple zero of the section $(f_1, \ldots, f_n)$. By a comment in [37, p. 17], the local degree $\deg_p(f_1, \ldots, f_n)$ at a simple zero p is determined by the Jacobian of $f_1, \ldots, f_n$ (after locally trivializing). We make this precise below.

Lemma 3.23. *Let $f_1, \ldots, f_n$ be hypersurfaces of $\mathbb{P}^n$ that intersect transversely at a point $p \in U_\ell$. To simplify notation, write $f_i^\ell := f_i \circ \tilde{\varphi}_\ell^{-1} = f_i((-1)^\ell \cdot \frac{x_0}{x_\ell}, \ldots, \frac{x_n}{x_\ell})$. Let*

$$J_\ell = \det\left(\tfrac{\partial f_i^\ell}{\partial(x_j/x_\ell)}\right)_{j \neq \ell},$$

and let $\mathrm{Tr}_{k(p)/k} : \mathrm{GW}(k(p)) \to \mathrm{GW}(k)$ be the trace on bilinear forms obtained by post-composing with the field trace $k(p) \to k$. Then

$$\deg_p(f_1, \ldots, f_n) = \mathrm{Tr}_{k(p)/k}\langle J_\ell(\tilde{\varphi}_\ell(p))\rangle.$$

Proof. By Lemma 3.12 and Lemma 3.13, the local trivialization $(f_1^\ell, \ldots, f_n^\ell)$ of our section $(f_1, \ldots, f_n)$ is compatible with our chosen relative orientation (relative to the effective Cartier divisor $D = \{x_0 = 0\}$) and Nisnevich coordinates $\{(U_i, \tilde{\varphi}_i)\}$. Since p is a simple zero of $(f_1, \ldots, f_n)$, [37, p. 17] gives us that

$$\deg_p(f_1, \ldots, f_n) = \mathrm{Tr}_{k(p)/k}\langle J_\ell(\tilde{\varphi}_\ell(p))\rangle,$$

as desired. $\square$

Moving forward, we will write $J_\ell(p)$ instead of $J_\ell(\tilde{\varphi}_\ell(p))$.

To obtain a geometric interpretation of the local degree at transverse intersections, we show how the Jacobian arises as a cross product of gradients. Working in one of our open affines, say $U_\ell \cong \mathbb{A}_k^n$, we let e_j be the unit vector corresponding to the $\frac{x_j}{x_\ell}$-axis for all $j \neq \ell$. The gradient of f_i^ℓ is given by $\nabla f_i^\ell = \sum_{j \neq \ell} \frac{\partial f_i^\ell}{\partial(x_j/x_\ell)} \cdot e_j$.

Definition 3.24. Let $v_i = \sum_j a_{ij} \cdot e_j$ be a vector in $\mathbb{A}^n$, where $a_{ij} \in k$ and e_j is as above. We may consider $\mathbb{A}^n$ as a subspace of $\mathbb{A}^{n+1}$, with a new unit vector $e_{n+1} := e_1 \times \cdots \times e_n$ corresponding to the direction perpendicular to $e_1, \ldots, e_n$. The *(n-ary) cross product* of $v_1, \ldots, v_n$ is a vector in the direction of e_{n+1} given by

$$\bigtimes_{i=1}^{n} v_i = \det \begin{pmatrix} a_{11} & \cdots & a_{1n} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & 0 \\ e_1 & \cdots & e_n & e_{n+1} \end{pmatrix}.$$

The dot product $(\bigtimes_{i=1}^{n} v_i) \cdot e_{n+1}$ is the signed volume of the parallelpiped bounded by $v_1, \ldots, v_n$. Note that this definition agrees with the more familiar notion of the cross product on $\mathbb{R}^3$.

Under this definition, the Jacobian $J_\ell(p)$ is the value of the dot product

$$\Big(\bigtimes_{i \neq \ell} (\nabla f_i^\ell(p))\Big) \cdot e_{n+1},$$

where $f_i^\ell(p) = f_i(\tilde{\varphi}_\ell(p))$. Thus the local degree at transverse intersections is described geometrically by the volume of the parallelpiped defined by the gradient vectors $\{\nabla f_i^\ell(p)\}_{i \neq \ell}$.

3.4.3 Non-transverse intersections

When our hypersurfaces $f_1, \ldots, f_n$ do not intersect transversely, the gradient directions of some f_i and f_j coincide. As a result, the parallelpiped spanned by the gradients of $f_1, \ldots, f_n$ has volume 0, so our previous geometric interpretation of $\deg_p(f_1, \ldots, f_n)$ no longer makes sense. We will discuss non-transverse intersections of pairs of curves in $\mathbb{P}^2$ and leave open the higher-dimensional case. Our goal is to reduce the calculation of the degree of two polynomials in two variables to the calculation of the degree of one power series in one variable. Let $f, g \in k[x, y]$ be polynomials of degrees c and d, respectively. For simplicity, we will assume that $p = (0, 0)$

is the origin and that $\frac{\partial g}{\partial y}(0,0) \neq 0$. We will also assume that $\operatorname{char} k = 0$, and we will discuss how to modify the following construction in positive characteristic.

Lemma 3.25. *Let f and g be curves in $\mathbb{P}^2$ that intersect at the origin, and assume that $\frac{\partial g}{\partial y}(0,0) \neq 0$. Then there exists some positive integer n and some $a_n \in k^\times$ such that*

$$\deg_0(f,g) = \begin{cases} \frac{n-1}{2} \cdot \mathbb{H} + \langle a_n \rangle & n \text{ odd,} \\ \frac{n}{2} \cdot \mathbb{H} & n \text{ even.} \end{cases}$$

Proof. By [36], the local degree $\deg_0(f,g)$ may be computed by the bilinear form constructed in [64]. We will refer to this bilinear form as the Scheja–Storch form. As discussed on [64, p. 178], the Scheja–Storch form constructed for the local ring $\frac{k[x,y]_0}{(f,g)}$ is isomorphic to the Scheja–Storch form for the completion $\frac{k[\![x,y]\!]}{(f,g)}$. We may thus work in $k[\![x,y]\!]$ in order to compute $\deg_0(f,g)$.

Note that if $a \in k^\times$, we have $\deg_0(f, ag) = \langle a \rangle \cdot \deg_0(f,g)$. We may thus scale g so that $\frac{\partial g}{\partial y}(0) = 1$. By Hensel's Lemma, there exists a power series $G(x) \in k[\![x]\!]$ such that $G(0) = 0$ and $g(x, G(x)) = 0$. We thus obtain an isomorphism

$$\begin{array}{ccc} \frac{k[\![x,y]\!]}{(f,g)} & \xrightarrow{\cong} & \frac{k[\![x]\!]}{(f(x,G(x)))} \\ y & \xmapsto{h} & G(x). \end{array}$$

Intuitively, the h transforms the curve $g(x,y)$ into our horizontal axis, and the curve $f(x, G(x))$ is the image of $f(x,y)$ under this transformation. In order for the isomorphism $h : \frac{k[\![x,y]\!]}{(f,g)} \xrightarrow{\cong} \frac{k[\![x]\!]}{(f(x,G(x)))}$ to preserve the local degree, it suffices to show that h sends $\operatorname{Jac}(f,g)|_{x=y=0}$ to $\operatorname{Jac}(f(x,G(x)))|_{x=0}$. Indeed, the local degree is determined by the Scheja–Storch form [37], and the Scheja–Storch form is determined by the Jacobian in characteristic 0 [64, (4.7) Korollar]. It follows that, given presentations of two local complete intersections $k[x_1, \ldots, x_m]_p/(r_1, \ldots, r_m)$ and $k[x_1, \ldots, x_n]_q/(s_1, \ldots, s_n)$ and an isomorphism $\phi : \frac{k[x_1,\ldots,x_m]_p}{(r_1,\ldots,r_m)} \to \frac{k[x_1,\ldots,x_n]_q}{(s_1,\ldots,s_n)}$, the bilinear form $\deg_p(r_1, \ldots, r_m)$ is isomorphic to the bilinear form $\deg_q(s_1, \ldots, s_n)$ if $\phi(\mathrm{Jac}(r_1, \ldots, r_m)(p)) = \mathrm{Jac}(s_1, \ldots, s_n)(q)$.

To show that the bilinear forms $\deg_0(f, g)$ and $\deg_0(f(x, G(x)))$ are isomorphic, we thus need h to send the Jacobian of f and g at $(0, 0)$ to the derivative of $f(x, G(x))$ at 0. That is, we need

$$\begin{aligned} h(\mathrm{Jac}(f,g))|_{x=y=0} &= h(\tfrac{\partial f}{\partial x} \cdot \tfrac{\partial g}{\partial y} - \tfrac{\partial f}{\partial y} \cdot \tfrac{\partial g}{\partial x})|_{x=y=0} \\ &= [f_x(x, G(x)) \cdot g_y(x, G(x)) - f_y(x, G(x)) \cdot g_x(x, G(x))]_{x=y=0} \\ &= f_x(0,0) \cdot g_y(0,0) - f_y(0,0) \cdot g_x(0,0) \end{aligned}$$

to be equal to

$$\begin{aligned} \frac{d}{dx} f(x, G(x))|_{x=0} &= [f_x(x, G(x)) + G'(x) \cdot f_y(x, G(x))]_{x=0} \\ &= f_x(0,0) + G'(0) \cdot f_y(0,0). \end{aligned}$$

Since $G(0) = 0$, we have $g_y(x, G(x))|_{x=0} = g_y(0,0)$, which is equal to 1 by assumption. By the chain rule, we have $0 = \frac{d}{dx} g(x, G(x)) = g_x(x, G(x)) + g_y(x, G(x)) \cdot G'(x)$, so $g_x(x, G(x)) = -g_y(x, G(x)) \cdot G'(x)$. Thus $g_x(x, G(x))|_{x=0} = -G'(0)$, so $f_x(0,0) \cdot g_y(0,0) - f_y(0,0) \cdot g_x(0,0) = f_x(0,0) + G'(0) \cdot f_y(0,0)$, as desired.

Writing $f(x, G(x)) = \sum_{i=n}^{\infty} a_i x^i = a_n x^n (1 + \sum_{i=1}^{\infty} b_i x^i)$ with $a_n \neq 0$, we note that $1 + \sum_{i=1}^{\infty} b_i x^i$ is a unit in $k[\![x]\!]$ and hence the Scheja–Storch form of $\frac{k[\![x]\!]}{(f(x,G(x)))}$ is equal to that of $\frac{k[\![x]\!]}{(a_n x^n)}$. We thus have that $\deg_0(f, g) = \deg_0(a_n x^n)$, which is given by $\frac{n-1}{2} \cdot \mathbb{H} + \langle a_n \rangle$ if n is odd and $\frac{n}{2} \cdot \mathbb{H}$ if n is even. $\square$

Remark 3.26. In any characteristic, the Scheja–Storch form is determined by a distinguished generator E of the socle of $\frac{k[x,y]_0}{(f,g)}$. In order to modify the previous argument for the positive characteristic case, one would need to ensure that $q \circ h$ sends E to the distinguished socle generator of $\frac{k[\![x]\!]}{(f(x,G(x)))}$.

Proposition 3.27. *We have that $n = i_0(f, g)$.*

Proof. By the above remarks and Proposition 3.20, both n and $i_0(f,g)$ are equal to the rank of $\deg_0(f,g)$. □

This proposition allows us to completely understand the local degree $\deg_0(f,g)$ when f and g meet at the origin with even multiplicity. When f and g intersect with odd multiplicity, it remains to study the term $\langle a_n \rangle$. We discuss the geometric interpretation of a_n over $\mathbb{R}$ in Lemma 3.31. We also give a recursive description of a_n in terms of the coefficients of f and g. Let

$$f = \sum_{i+j=0}^{c} f_{i,j} x^i y^j \qquad \text{and} \qquad g = \sum_{i+j=0}^{d} g_{i,j} x^i y^j.$$

We compute a_n as the coefficient of $x^n = x^{i_0(f,g)}$ in $f(x, G(x))$. Thus $a_n = \sum_{i+j=n} f_{i,j} \cdot \gamma(j)$, where $\gamma(j)$ is the coefficient of x^j in $G(x)^j$. But $\gamma(j)$ is equal to the coefficient of x^j in $(G_0 + G_1 x + \ldots + G_j x^j)^j$, where $G(x) = \sum_{i=0}^{\infty} G_i x^i$. Expanding this product, we see that

$$\gamma(j) = \sum_{\substack{t_0+\ldots+t_j=j \\ \sum_u u t_u = j}} \binom{j}{t_0, \ldots, t_j} \prod_{u=0}^{j} G_u^{t_u},$$

where $\binom{j}{t_0,\ldots,t_j}$ denotes the multinomial coefficient. All that remains is to determine the coefficients G_i of the power series $G(x)$. This is accomplished by repeatedly taking implicit derivatives. By assumption, $g(0,0) = 0$, so we have $G_0 = 0$. Next, evaluating the partial derivative $\frac{\partial g}{\partial x} = \sum_{i+j=0}^{d} g_{i,j}(i x^{i-1} y^j + j x^i y^{j-1} \cdot \frac{\partial y}{\partial x}) = 0$ at $(0,0)$ gives us that $G_1 = \frac{\partial y}{\partial x}(0,0) = -g_{1,0}/g_{0,1}$. Iterating this process allows us to compute $G_i = \frac{1}{i!} \cdot \frac{\partial^i y}{\partial x^i}(0,0)$.

Remark 3.28. In positive characteristic, we need an alternative form of the derivative in order to write down a Taylor series $G(x)$ under Hensel's Lemma. The Hasse derivative [32, p. 64] should be suitable for this purpose.

3.5 Specializations over some specific fields

For the following discussion, we let $N = -n - 1 + \sum_{i=1}^{n} d_i$ and assume that $N \equiv 0 \mod 2$ so that $\mathcal{O}_{d_1,\ldots,d_n} \to \mathbb{P}^n$ is relatively orientable. Throughout this article, we have generally assumed that all intersections of $f_1, \ldots, f_n$ are transverse. This allows us to give better geometric interpretations of $\deg_p(f_1, \ldots, f_n)$. However, we also address non-transverse intersections in Sections 3.5.1 and 3.5.2. Our approach is as follows. For any field k, taking the rank of bilinear forms gives a homomorphism $\mathrm{GW}(k) \to \mathbb{Z}$. When k is algebraically closed, $\mathrm{rank} : \mathrm{GW}(k) \xrightarrow{\cong} \mathbb{Z}$ is an isomorphism. When k is not algebraically closed, we apply some other invariant to get a homomorphism of the form $\mathrm{rank} \times \mathrm{invariant} : \mathrm{GW}(k) \to \mathbb{Z} \times G$ for some group G. The spirit of $\mathbb{A}^1$-enumerative geometry is that the $\mathbb{Z}$-valued count coming from the rank describes the geometric phenomena of classical enumerative geometry, while the additional G-valued count coming from the other invariant carries extra arithmetic-geometric information. This extra arithmetic-geometric information enriches the classical enumerative theorem when we work over a non-algebraically closed field.

3.5.1 Bézout's theorem over $\mathbb{C}$

Since $\mathbb{C}$ is algebraically closed, $\mathrm{rank} : \mathrm{GW}(\mathbb{C}) \xrightarrow{\cong} \mathbb{Z}$ is an isomorphism. We thus recover Bézout's theorem over $\mathbb{C}$ by applying rank to both sides of Equation 3.2. We know that $\mathrm{rank}\, \mathbb{H} = 2$, so $\mathrm{rank}\, e(\mathcal{O}_{d_1,\ldots,d_n}) = d_1 \cdots d_n$. Moreover, we have $\mathrm{rank} \deg_p(f_1, \ldots, f_n) = i_p(f_1, \ldots, f_n)$ by Proposition 3.20. This gives us Equation 3.1, as expected.

3.5.2 Bézout's theorem over $\mathbb{R}$

We can represent any non-degenerate symmetric bilinear form over $\mathbb{R}$ by a diagonal matrix, where each diagonal entry is either $1, -1$, or 0. By Sylvester's law of inertia, the isomorphism class of a bilinear form over $\mathbb{R}$ is determined by its rank and its signature, which we define to be the difference between the number of 1s and the number of -1s on the diagonal. We thus have an isomorphism $\mathrm{GW}(\mathbb{R}) \cong \mathbb{Z} \times \mathbb{Z}$ induced by $\mathrm{rank} \times \mathrm{sign} : \mathrm{GW}(\mathbb{R}) \to \mathbb{Z} \times \mathbb{Z}$.

Remark 3.29. Since $|\,\mathrm{sign}(\beta)| \leqslant \mathrm{rank}(\beta)$, the homomorphism

$$\mathrm{rank} \times \mathrm{sign} : \mathrm{GW}(\mathbb{R}) \to \mathbb{Z} \times \mathbb{Z}$$

is not surjective. However, there is a group isomorphism $\mathrm{GW}(\mathbb{R}) \cong \mathbb{Z}\times\mathbb{Z}$ [39, Chapter II, Theorem 3.2 (4)]. Since $\mathrm{rank} \times \mathrm{sign}$ is injective, it follows that the image of $\mathrm{rank} \times \mathrm{sign}$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

We obtain Bézout's theorem over $\mathbb{R}$ by applying sign to both sides of Equation 3.2. Since $\mathrm{sign}\,\mathbb{H} = 0$, we have that $\mathrm{sign}\, e(\mathcal{O}_{d_1,\ldots,d_n}) = 0$. When $f_1, \ldots, f_n$ intersect transversely at $p \in U_\ell$, the signature of $\deg_p(f_1, \ldots, f_n)$ is given by the sign of the volume of the parallelpiped defined by the gradient vectors $\{\nabla f_i^\ell(p)\}_{i\neq\ell}$.

Remark 3.30. If p is a non-rational intersection point of $f_1, \ldots, f_n$, then

$$\mathrm{sign}\deg_p(f_1, \ldots, f_n) = 0.$$

To see this, suppose $\deg_p(f_1, \ldots, f_n) = \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}\langle a + ib\rangle$. Since every element of $\mathbb{C}$ is a square, we have $\langle a + ib\rangle = \langle 1\rangle$. This bilinear form can thus be represented by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which indeed has signature zero.

By Remark 3.30, the local degree at non-real zeros does not contribute anything to our overall signed count $\mathrm{sign}\, e(\mathcal{O}_{d_1,\ldots,d_n})$. In particular, we only need to consider real zeros, so we may restrict our attention to the real points of $\mathbb{P}^n$ and the hypersurfaces $f_1, \ldots, f_n$. This allows us to apply Milnor's local alteration trick [50, §6, Step 2] from real differential topology.

Lemma 3.31. *Let $\sigma \in \mathcal{O}_{d_1,\ldots,d_n}$ be a section corresponding to the hypersurfaces $f_1, \ldots, f_n$ in $\mathbb{P}^n_{\mathbb{R}}$ that intersect at a rational point p, and let U be an open neighborhood (in the real topology) about p that does not contain any other zeros of σ. Then there exist open neighborhoods $U' \subseteq U$ about p and $V \subseteq H^0(\mathbb{P}^n_{\mathbb{R}}, \mathcal{O}_{d_1,\ldots,d_n})$ about σ such that* $\operatorname{sign}\deg_p \sigma = \sum_{q\in U'} \operatorname{sign}\deg_q \sigma'$ *for all $\sigma' \in V$.*

Proof. This follows from the proof of [40, Lemma 2.4]. We recall the relevant details for the reader's convenience. Let $\varphi : \mathcal{O}_{d_1,\ldots,d_n}|_U \xrightarrow{\cong} \mathbb{R}^n$ and $\psi : \mathcal{T}U \xrightarrow{\cong} \mathbb{R}^n$ be local trivializations such that $\det(\varphi^{-1} \circ \psi)$ is a square in $\mathcal{O}(-n-1+\sum_{i=1}^n d_i)$ under the chosen relative orientation. On U, we can equate sections $\sigma \in \mathcal{O}_{d_1,\ldots,d_n}|_U$ with vector fields $v_\sigma = \psi^{-1} \circ \varphi(\sigma)$ on U, which allows us to use Milnor's local alteration trick as follows. Let $U'' \subset U' \subset U$ be sufficiently small (real) open neighborhoods about p, and let $\lambda : U \to [0,1]$ be a smooth bump function such that $\lambda|_{U''} = 1$ and $\lambda|_{U\setminus U'} = 0$. Taking a sufficiently small regular value y of v_σ, the vector field $v(x) = v_\sigma(x) - \lambda(x)y$ is non-degenerate on U'. Let $\iota_q w$ denote Milnor's local index of a vector field w at a zero q. By [50, §6, Theorem 1], we have that $\sum_{q\in U'} \iota_q v$ is equal to the degree of the Gauss map $\bar{v} : \partial U' \to S^{n-1}$. The degree of the Gauss map, and hence the sum $\sum_{q\in U'} \iota_q v$, is continuous (and thus locally constant) in σ. Let $V \subseteq H^0(\mathbb{P}^n_{\mathbb{R}}, \mathcal{O}_{d_1,\ldots,d_n})$ be a sufficiently small neighborhood about σ; in particular, the zeros of the altered vector field v' corresponding to the section σ' should remain in U' as σ' varies. Then for all $\sigma' \in V$, we have that $\iota_p v = \sum_{q\in U'} \iota_q v'$. Finally, we remark that $\iota_p v = \operatorname{sign}\deg_p \sigma$, as was proved by Eisenbud and Levine [24, Main Theorem]. $\square$

As an aside, this proof implies that the degree of the Gauss map associated to the hypersurfaces $f_1, \ldots, f_n$ at an intersection point p is bounded by the intersection multiplicity $i_p(f_1, \ldots, f_n)$. We state this as a proposition.

Proposition 3.32. *Let σ and v be as in Lemma 3.31, and let $\bar{v}$ be the corresponding*

Gauss map. Then $|\deg \bar{v}| \leqslant i_p(f_1, \ldots, f_n)$.

Proof. By [50, §6, Theorem 1] and [24, Main Theorem], we have that $\deg \bar{v} = \operatorname{sign} \deg_p \sigma$. Since $|\operatorname{sign} \deg_p \sigma| \leqslant \operatorname{rank} \deg_p \sigma$, Proposition 3.20 implies that $|\deg \bar{v}| \leqslant i_p(f_1, \ldots, f_n)$. □

Remark 3.33. By Lemma 3.31, we can compute the crossing sign of a non-transverse intersection by slightly perturbing our chosen section. Since generic intersections are transverse, we may choose our new section to have only transverse intersections. As a consequence, the crossing sign of a non-transverse intersection is given by a sum of crossing signs of transverse intersections. This is illustrated in Figure 3.2.

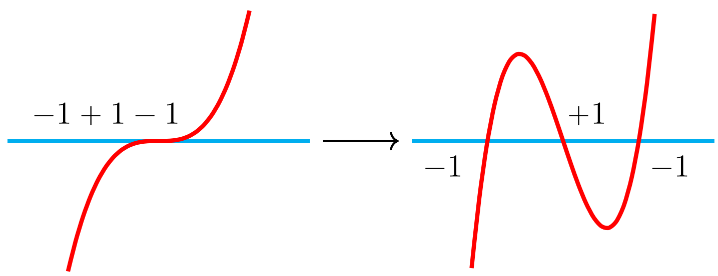

Figure 3.2: The crossing sign at a non-transverse intersection.

Perturbing our hypersurfaces to ensure that they intersect transversely, we may thus call $\operatorname{sign} \deg_p(f_1, \ldots, f_n)$ the *crossing sign* of $f_1, \ldots, f_n$ at p, and we obtain the following theorem.

Theorem 3.34 (Bézout's theorem over $\mathbb{R}$)**.** *Let* $f_1, \ldots, f_n$ *be hypersurfaces in* $\mathbb{P}^n_{\mathbb{R}}$*, and let* d_i *be the degree of* f_i *for each* i*. Assume that* $f_1, \ldots, f_n$ *have no common components and that* $-n-1+\sum_{i=1}^n d_i \equiv 0 \mod 2$*. Then, summing over the intersection points of* $f_1, \ldots, f_n$*, there are an equal number of positive and negative crossings of* $f_1, \ldots, f_n$*.*

Example 3.35. We can now make sense of the problematic conic and cubic in $\mathbb{P}^2_{\mathbb{R}}$ from Figure 3.1. To be specific, let $f_1 = x_0^2 x_2 - x_1^3$ and $f_2 = x_1^2 + x_2^2 - 2x_0^2$. The only intersection points of f_1 and f_2 are $p_1 = [1:-1:-1]$ and $p_2 = [1:1:1]$. The crossing sign of f_1 and f_2 at p_i is given by the right hand rule on the gradient vectors of f_1^0 and f_2^0 at $\varphi_0(p_i)$. We now demonstrate this calculation. Let e_i be the the unit basis vector along the $(\frac{x_i}{x_0})$-axis, and let $e_3 := e_1 \times e_2$. Then $\nabla f_1^0 = -3(\frac{x_1}{x_0})^2 \cdot e_1 + e_2$ and $\nabla f_2^0 = 2(\frac{x_1}{x_0}) \cdot e_1 + 2(\frac{x_2}{x_0}) \cdot e_2$, so $\nabla f_1^0 \times \nabla f_2^0 = (-6(\frac{x_1}{x_0})^2(\frac{x_2}{x_0}) - 2(\frac{x_1}{x_0})) \cdot e_3$. The crossing sign at p_i is computed by taking the sign of the dot product of $\nabla f_1^0 \times \nabla f_2^0(\varphi_0(p_i))$ and e_3. This gives us

$$\operatorname{sign}(\nabla f_1^0 \times \nabla f_2^0(\varphi(p_1)) \cdot e_3) = \operatorname{sign}(6+2) = 1,$$

$$\operatorname{sign}(\nabla f_1^0 \times \nabla f_2^0(\varphi(p_2)) \cdot e_3) = \operatorname{sign}(-6-2) = -1.$$

We illustrate this calculation in Figure 3.3.

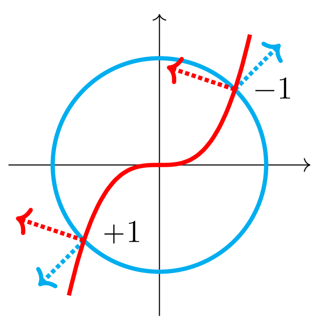

Figure 3.3: Signed intersections over $\mathbb{R}$.

3.5.3 Bézout's theorem over finite fields

Let $\mathbb{F}_q$ be the finite field of order q. Over $\mathbb{F}_q$, non-degenerate symmetric bilinear forms are classified up to isomorphism by their rank and discriminant [39, Chapter II, Theorem 3.5 (1)], so we have an isomorphism $\operatorname{rank} \times \operatorname{disc} : \mathrm{GW}(\mathbb{F}_q) \xrightarrow{\cong} \mathbb{Z} \times \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2$. For simplicity, we assume that $\mathbb{F}_q$ has odd characteristic, so that q is the power of some odd prime. As a result, we have $\mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2 \cong \mathbb{Z}/2\mathbb{Z}$, and we can classify elements of a given rank in $\mathrm{GW}(\mathbb{F}_q)$ by whether or not their discriminant is a square. Taking the discriminant of Equation 3.2, we get $\operatorname{disc} e(\mathcal{O}_{d_1,\ldots,d_n}) = (-1)^{d_1 \cdots d_n/2}$. This makes sense, since the fraction $\frac{d_1 \cdots d_n}{2}$ is an integer by Remark 3.5. We note that $\operatorname{disc} e(\mathcal{O}_{d_1,\ldots,d_n})$ is a perfect square in $\mathbb{F}_q$ if and only if $\frac{d_1 \cdots d_n}{2}$ is even or $q \equiv 1 \mod 4$. The left hand side of Equation 3.2 gives us

$$\prod_{\text{points}} \operatorname{disc} \deg_p(f_1, \ldots, f_n).$$

This product is a perfect square in $\mathbb{F}_q$ if and only if there are an even number of intersection points p such that $\operatorname{disc} \deg_p(f_1, \ldots, f_n)$ is not a square. It thus remains to determine when $\operatorname{disc} \deg_p(f_1, \ldots, f_n)$ is a square in $\mathbb{F}_q$. Let $\mathbb{F}_{q^b}$ be the field of definition of a transverse intersection point $p \in U_\ell$. By [21, Section II.2], we have that $\operatorname{disc} \operatorname{Tr}_{\mathbb{F}_{q^b}/\mathbb{F}_q}\langle J_\ell(p)\rangle = \operatorname{norm}(J_\ell(p)) \cdot \operatorname{disc} \operatorname{Tr}_{\mathbb{F}_{q^b}/\mathbb{F}_q}\langle 1\rangle$. Since $\operatorname{norm} : \mathbb{F}_{q^b}^\times \to \mathbb{F}_q^\times$ is a homomorphism, norm takes squares in $\mathbb{F}_{q^b}^\times$ to squares in $\mathbb{F}_q^\times$. On the other hand, Hilbert's Theorem 90 implies that $\operatorname{norm} : \mathbb{F}_{q^b}^\times \to \mathbb{F}_q^\times$ is surjective, so if $\operatorname{norm}(J_\ell(p))$ is a square, then we have $\operatorname{norm}(J_\ell(p)) = \operatorname{norm}(y^2)$ for some $y \in \mathbb{F}_{q^b}^\times$. Hilbert's Theorem 90 also implies that norm has kernel $\{z^{q-1} : z \in \mathbb{F}_{q^b}^\times\}$, so there exists some $z \in \mathbb{F}_{q^b}^\times$ such that $J_\ell(p) = y^2 z^{q-1}$. It follows that $J_\ell(p) = (yz^{(q-1)/2})^2$, so $J_\ell(p)$ is a square if and only if $\operatorname{norm}(J_\ell(p))$ is a square.

Definition 3.36. We will call a transverse intersection point p of $f_1, \ldots, f_n$ a *positive* intersection point if $J_\ell(p)$ is a square in $\mathbb{F}_{q^b}^\times$. We call p a *negative* intersection point if $J_\ell(p)$ is not a square in $\mathbb{F}_{q^b}^\times$.

To characterize when $\operatorname{Tr}_{\mathbb{F}_{q^b}/\mathbb{F}_q}\langle 1\rangle$ is a square, we let α be a primitive element of the extension $\mathbb{F}_{q^b}/\mathbb{F}_q$, so that $\{1, \alpha, \ldots, \alpha^{b-1}\}$ is an $\mathbb{F}_q$-basis for $\mathbb{F}_{q^b}$. The matrix representing $\mathrm{Tr}_{\mathbb{F}_{q^b}/\mathbb{F}_q}\langle 1\rangle$ with respect to this basis is MM^T, where

$$M = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha & \alpha^q & \cdots & \alpha^{q^{b-1}} \\ \alpha^2 & \alpha^{2q} & \cdots & \alpha^{2q^{b-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha^{b-1} & \alpha^{(b-1)q} & \cdots & \alpha^{(b-1)q^{b-1}} \end{pmatrix}.$$

Indeed, the $(i,j)^{\text{th}}$ entry of MM^T is

$$\begin{aligned} \sum_{\ell=0}^{b-1} \alpha^{(i-1)q^\ell}\alpha^{(j-1)q^\ell} &= \sum_{\ell=0}^{b-1} (\alpha^{i+j-2})^{q^\ell} \\ &= \mathrm{Tr}_{\mathbb{F}_{q^b}/\mathbb{F}_q}(\alpha^{i+j-2}). \end{aligned}$$

By definition of the trace form, this is equal to the $(i,j)^{\text{th}}$ entry of the Gram matrix of $\mathrm{Tr}_{\mathbb{F}_{q^b}/\mathbb{F}_q}\langle 1\rangle$ with respect to the basis $\{1, \alpha, \dots, \alpha^{b-1}\}$. Thus we have

$$\begin{aligned} \operatorname{disc} \mathrm{Tr}_{\mathbb{F}_{q^b}/\mathbb{F}_q}\langle 1\rangle &= \det(MM^T) = (\det M)^2 \\ &= \prod_{0\leqslant i<j\leqslant b-1} (\alpha^{q^j} - \alpha^{q^i})^2. \end{aligned}$$

So $\operatorname{disc} \mathrm{Tr}_{\mathbb{F}_{q^b}/\mathbb{F}_q}\langle 1\rangle$ is a square in $\mathbb{F}_q$ if and only if $\delta = \prod_{i<j}(\alpha^{q^j} - \alpha^{q^i})$ is an element of $\mathbb{F}_q$. Since $\mathrm{Gal}(\mathbb{F}_{q^b}/\mathbb{F}_q)$ is cyclic and generated by the Frobenius F, we have that $\delta \in \mathbb{F}_q$ if and only if $F(\delta) = \delta$. But $F(\delta) = \varepsilon\delta$, where ε is the sign of the permutation $(1\ 2\ \dots\ b-1)$. We know that $\varepsilon = 1$ if b is odd and $\varepsilon = -1$ if b is even, so $\operatorname{disc} \mathrm{Tr}_{\mathbb{F}_{q^b}/\mathbb{F}_q}\langle 1\rangle$ is a square in $\mathbb{F}_q$ if and only if b is odd. Summing all this together, we have

$$\operatorname{disc} \mathrm{Tr}_{\mathbb{F}_{q^b}/\mathbb{F}_q}\langle J_\ell(p)\rangle = \begin{cases} \text{a square} & \text{if } p \text{ is a positive intersection and } b \text{ is odd,} \\ \text{a square} & \text{if } p \text{ is a negative intersection and } b \text{ is even,} \\ \text{a non-square} & \text{if } p \text{ is a positive intersection and } b \text{ is even,} \\ \text{a non-square} & \text{if } p \text{ is a negative intersection and } b \text{ is odd.} \end{cases}$$

We summarize this information in the following theorem.

Theorem 3.37 (Bézout's theorem over $\mathbb{F}_q$)**.** *Let $f_1, \ldots, f_n$ be hypersurfaces in $\mathbb{P}^n_{\mathbb{F}_q}$, and let d_i be the degree of f_i for each i. Assume that $f_1, \ldots, f_n$ intersect transversely and that $-n-1+\sum_{i=1}^n d_i \equiv 0 \mod 2$.*

(a) If $\frac{d_1\cdots d_n}{2}$ is even or $q \equiv 1 \mod 4$, then

$$\begin{aligned} &\#\ \textit{positive intersections with field of definition } \mathbb{F}_{q^b} \textit{ for } b \textit{ even} \\ +\ &\#\ \textit{negative intersections with field of definition } \mathbb{F}_{q^b} \textit{ for } b \textit{ odd} \equiv 0 \mod 2. \end{aligned}$$

(b) If $\frac{d_1\cdots d_n}{2}$ is odd and $q \not\equiv 1 \mod 4$, then

$$\begin{aligned} &\#\ \textit{positive intersections with field of definition } \mathbb{F}_{q^b} \textit{ for } b \textit{ even} \\ +\ &\#\ \textit{negative intersections with field of definition } \mathbb{F}_{q^b} \textit{ for } b \textit{ odd} \equiv 1 \mod 2. \end{aligned}$$

3.5.4 Bézout's theorem over $\mathbb{C}((t))$

We begin by describing $\mathrm{GW}(\mathbb{C}((t)))$. The field $\mathbb{C}((t))$ of Laurent series consists of elements of the form $g = \sum_{i=m}^\infty a_i t^i$, where $m \in \mathbb{Z}$ and $a_m \neq 0$, and of the element 0. With the valuation $v(g) = m$, the pair $(\mathbb{C}((t)), v)$ is a *complete discretely valuated field* (see e.g. [39, Chapter VI, Section 1]). By slight abuse of terminology, we call the non-zero elements of $\mathbb{C}((t))$ with $v(g) = 0$ *units*. Units in $\mathbb{C}((t))$ are of the form $g = \sum_{i=0}^\infty a_i t^i$. Fixing t as our uniformizer, every non-zero element of $\mathbb{C}((t))$ can be written as $g = ut^{v(g)}$ for some unit u. By relation (i) of Section 1.2.1, it follows that $\langle g\rangle = \langle u\rangle$ if $v(g)$ is even and $\langle g\rangle = \langle ut\rangle$ if $v(g)$ is odd. By [39, Chapter VI, Lemma 1.1], a unit u is a square in $\mathbb{C}((t))$ if and only if $u(0) = a_0$ is a square in $\mathbb{C}$. Since $\mathbb{C}$ is algebraically closed, we have $\langle g\rangle = \langle 1\rangle$ if $v(g)$ is even and $\langle g\rangle = \langle t\rangle$ if $v(g)$ is odd. It follows that $\mathrm{GW}(\mathbb{C}((t)))$ is generated by $\langle 1\rangle$ and $\langle t\rangle$. Since $\langle 1\rangle = \langle -1\rangle$ and $\langle t\rangle = \langle -t\rangle$, we have that $2\langle 1\rangle = 2\langle t\rangle = \mathbb{H}$. We thus get a well-defined group isomorphism

$$\mathrm{GW}(\mathbb{C}((t))) \cong \frac{\mathbb{Z}[\langle t\rangle]}{(\langle t\rangle^2 - 1, 2\langle t\rangle - 2)} \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

We would like to realize the isomorphism $\mathrm{GW}(\mathbb{C}(\!(t)\!)) \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ as a map of the form rank $\times$ invariant.

Proposition 3.38. *Let* $\operatorname{rank}(m\langle 1\rangle + n\langle t\rangle) = m + n$*, and let* $\operatorname{disc}(m\langle 1\rangle + n\langle t\rangle) = n \mod 2$*. Then* $\operatorname{rank} \times \operatorname{disc} : \mathrm{GW}(\mathbb{C}(\!(t)\!)) \xrightarrow{\cong} \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ *is an isomorphism of abelian groups.*

Proof. It can readily be checked that rank $\times$ disc is a well-defined group homomorphism. Moreover, rank $\times$ disc is surjective, since $\operatorname{rank} \times \operatorname{disc}(m\langle 1\rangle) = (m, 0)$ and $\operatorname{rank} \times \operatorname{disc}((m-1)\langle 1\rangle + \langle t\rangle) = (m, 1)$ for all $m \in \mathbb{Z}$. Finally, *rank* $\times$ disc is injective. Indeed, if $\operatorname{rank} \times \operatorname{disc}(m\langle 1\rangle + n\langle t\rangle) = (0, 0)$, then $m + n = 0$ and $n \equiv 0 \mod 2$. Thus $m = -n = -2s$ for some $s \in \mathbb{Z}$, so $m\langle 1\rangle + n\langle t\rangle = s(2\langle t\rangle - 2\langle 1\rangle) = 0$. $\square$

Remark 3.39. The homomorphism $\operatorname{disc} : \mathrm{GW}(\mathbb{C}(\!(t)\!)) \to \mathbb{Z}/2\mathbb{Z}$ is the traditional discriminant, simply written additively.

We may now obtain Bézout's theorem over $\mathbb{C}(\!(t)\!)$ by applying disc to Equation 3.2. Since $\mathbb{H} = 2 \cdot \langle 1\rangle = 2 \cdot \langle t\rangle$, we have $\operatorname{disc} \mathbb{H} = 0$ and hence $\operatorname{disc} e(\mathcal{O}_{d_1,\ldots,d_n}) = 0$. This will be equal to $\operatorname{disc}(\sum \deg_p(f_1, \ldots, f_n)) = \sum \operatorname{disc} \deg_p(f_1, \ldots, f_n)$, so we need to understand $\operatorname{disc} \deg_p(f_1, \ldots, f_n)$. Any degree m extension of $\mathbb{C}(\!(t)\!)$ is a cyclic Galois extension of the form $\mathbb{C}(\!(t^{1/m})\!)$ [60, XIII.2, p. 191]. By our previous discussion, $\mathrm{GW}(\mathbb{C}(\!(t^{1/m})\!))$ is generated by $\langle 1\rangle$ and $\langle t^{1/m}\rangle$, so at transverse intersection points, $\deg_p(f_1, \ldots, f_n)$ is either of the form $\operatorname{Tr}_{\mathbb{C}(\!(t^{1/m})\!)/\mathbb{C}(\!(t)\!)}\langle 1\rangle$ or $\operatorname{Tr}_{\mathbb{C}(\!(t^{1/m})\!)/\mathbb{C}(\!(t)\!)}\langle t^{1/m}\rangle$.

Definition 3.40. In analogy with the finite field case, we call a transverse intersection point p of $f_1, \ldots, f_n$ a *positive* intersection point if $\deg_p(f_1, \ldots, f_n) = \operatorname{Tr}_{\mathbb{C}(\!(t^{1/m})\!)/\mathbb{C}(\!(t)\!)}\langle 1\rangle$. Similarly, e call p a *negative* intersection point if $\deg_p(f_1, \ldots, f_n) = \operatorname{Tr}_{\mathbb{C}(\!(t^{1/m})\!)/\mathbb{C}(\!(t)\!)}\langle t^{1/m}\rangle$.

Lemma 3.41. *If m is a positive integer, then we have* $\operatorname{disc} \operatorname{Tr}_{\mathbb{C}(\!(t^{1/m})\!)/\mathbb{C}(\!(t)\!)}\langle 1\rangle \equiv m - 1 \mod 2$ *and* $\operatorname{disc} \operatorname{Tr}_{\mathbb{C}(\!(t^{1/m})\!)/\mathbb{C}(\!(t)\!)}\langle t^{1/m}\rangle \equiv m \mod 2$.

Proof. Mirroring the case of finite fields, we let $t^{1/m}$ be our primitive element. The Galois group of $\mathbb{C}(\!(t^{1/m})\!)$ over $\mathbb{C}(\!(t)\!)$ is generated by $\varphi : t^{1/m} \mapsto \zeta t^{1/m}$, where $\zeta = e^{2\pi i/m}$ is a primitive m^{th} root of unity. We have the $\mathbb{C}(\!(t)\!)$-basis $\{1, t^{1/m}, \ldots, t^{(m-1)/m}\}$ for $\mathbb{C}(\!(t^{1/m})\!)$. The Gram matrix for $\mathrm{Tr}_{\mathbb{C}(\!(t^{1/m})\!)/\mathbb{C}(\!(t)\!)}\langle u \rangle$ with respect to this basis is given by the product AB, where $a_{ij} = \varphi^{j-1}(t^{(i-1)/m})$ and $b_{ij} = \varphi^{i-1}(ut^{(j-1)/m})$ are the entries in the i^{th} row and j^{th} column of A and B, respectively. When $u = 1$, this product of matrices has entries

$$c_{ij} = t^{(i+j-2)/m} \sum_{\ell=0}^{m-1} \zeta^{\ell(i+j-2)} = \begin{cases} 0 & m \nmid i+j-2, \\ mt^{(i+j-2)/m} & m \mid i+j-2. \end{cases}$$

As a result, we have

$$\mathrm{Tr}_{\mathbb{C}(\!(t^{1/m})\!)/\mathbb{C}(\!(t)\!)}\langle 1 \rangle = \begin{pmatrix} m & 0 & \cdots & 0 \\ 0 & 0 & \cdots & mt \\ \vdots & \vdots & \iddots & \vdots \\ 0 & mt & \cdots & 0 \end{pmatrix} = (m-1) \cdot \langle t \rangle + \langle 1 \rangle.$$

Thus $\operatorname{disc} \mathrm{Tr}_{\mathbb{C}(\!(t^{1/m})\!)/\mathbb{C}(\!(t)\!)}\langle 1 \rangle \equiv m - 1 \mod 2$. When $u = t^{1/m}$, the product AB has entries

$$c_{ij} = t^{(i+j-1)/m} \sum_{\ell=0}^{m-1} \zeta^{\ell(i+j-1)} = \begin{cases} 0 & m \nmid i+j-1, \\ mt^{(i+j-1)/m} & m \mid i+j-1. \end{cases}$$

As a result, we have

$$\mathrm{Tr}_{\mathbb{C}(\!(t^{1/m})\!)/\mathbb{C}(\!(t)\!)}\langle t^{1/m} \rangle = \begin{pmatrix} 0 & \cdots & 0 & mt \\ 0 & \cdots & mt & 0 \\ \vdots & \iddots & \vdots & \vdots \\ mt & \cdots & 0 & 0 \end{pmatrix} = m \cdot \langle t \rangle.$$

Thus $\operatorname{disc} \mathrm{Tr}_{\mathbb{C}(\!(t^{1/m})\!)/\mathbb{C}(\!(t)\!)}\langle t^{1/m} \rangle \equiv m \mod 2$. □

As a result, we have proved the following theorem.

Theorem 3.42 (Bézout's theorem over $\mathbb{C}((t))$). *Let $f_1, \ldots, f_n$ be hypersurfaces in $\mathbb{P}^n_{\mathbb{C}((t))}$, and let d_i be the degree of f_i for each i. Assume that $f_1, \ldots, f_n$ intersect transversely and that $-n-1+\sum_{i=1}^n d_i \equiv 0 \mod 2$. Then*

$$\begin{aligned} &\#\textit{ positive intersections with field of definition } \mathbb{C}((t^{1/m})) \textit{ for } m \textit{ even} \\ &+ \#\textit{ negative intersections with field of definition } \mathbb{C}((t^{1/m})) \textit{ for } m \textit{ odd} \equiv 0 \mod 2. \end{aligned}$$

3.5.5 Bézout's theorem over $\mathbb{Q}$

In contrast with the previous fields we have considered, we need several invariants to understand $\mathrm{GW}(\mathbb{Q})$. Letting $\mathrm{W}(\mathbb{Q})$ denote the Witt ring of $\mathbb{Q}$, we have

$$\begin{tikzcd} \mathrm{GW}(\mathbb{Q}) \arrow[r, "\mathrm{rank}"] \arrow[d, "\mathrm{mod}\ \mathbb{H}"'] & \mathbb{Z} \arrow[d, "\mathrm{mod}\ 2"] \\ \mathrm{W}(\mathbb{Q}) \arrow[r] & \mathbb{Z}/2\mathbb{Z}. \end{tikzcd}$$

This gives us an isomorphism $\mathrm{rank} \times \mathrm{mod}\ \mathbb{H} : \mathrm{GW}(\mathbb{Q}) \xrightarrow{\cong} \mathbb{Z} \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{W}(\mathbb{Q})$. In order to obtain Bézout's theorem over $\mathbb{Q}$, it thus suffices to describe $\mathrm{W}(\mathbb{Q})$. By the weak Hasse-Minkowski principle (see e.g. [39, Chapter VI, Section 4, (4.4)]), we have an isomorphism

$$\partial := \mathrm{sign} \times \partial_2 \times \oplus\partial_p : \mathrm{W}(\mathbb{Q}) \xrightarrow{\cong} \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \bigoplus_{p \neq 2} \mathrm{W}(\mathbb{F}_p).$$

When $p \equiv 3 \mod 4$, we have $\mathrm{W}(\mathbb{F}_p) \cong \mathbb{Z}/4\mathbb{Z}$, generated by $\langle 1 \rangle$. When $p \equiv 1 \mod 4$, the Witt ring $\mathrm{W}(\mathbb{F}_p)$ is isomorphic to the group ring $(\mathbb{Z}/2\mathbb{Z})[\mathbb{F}_p^\times/(\mathbb{F}_p^\times)^2]$, whose underlying group structure is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, generated by $\langle 1 \rangle$ and $\langle r \rangle$, where r is a non-square in $\mathbb{F}_p^\times$. The invariant ∂_2 is given by $\partial_2(\beta) = v_2(\mathrm{disc}\,\beta) \mod 2$, where v_2 is the 2-adic valuation. For any odd prime p, any element of $\mathbb{Q}$ may be written as $q = up^{v_p(q)}$, where v_p is the p-adic valuation and $v_p(u) = 0$. It follows that $\langle q \rangle = \langle u \rangle$ if $v_p(q)$ is even and $\langle q \rangle = \langle up \rangle$ if $v_p(q)$ is odd. We define ∂_p by setting

$\partial_p\langle u\rangle = 0$ for any p-adic unit u and $\partial_p\langle up\rangle = \langle\bar{u}\rangle$, where $\bar{u}$ is the image of u under the composition $\mathbb{Q} \hookrightarrow \mathbb{Q}_p \to \mathbb{F}_p$. Explicitly, if q is a non-zero rational number with $v_p(q)$ odd, then we write $q = \frac{a}{b} \cdot p^{v_p(q)}$ with a and b prime to p. Since $\langle q\rangle = \langle \frac{a}{b} \cdot p\rangle$ in $\mathrm{W}(\mathbb{Q})$, our definition for ∂_p gives us $\partial_p\langle q\rangle = \langle (a \mod p)(b \mod p)^{-1}\rangle$ in $\mathrm{W}(\mathbb{F}_p)$. We obtain Bézout's theorem over $\mathbb{Q}$ by applying $\partial \circ \bmod \mathbb{H}$ to both sides of Equation 3.2. Since $e(\mathcal{O}_{d_1,\ldots,d_n}) \in \mathrm{GW}(\mathbb{Q})$ is a multiple of $\mathbb{H}$, it has trivial image in $\mathrm{W}(\mathbb{Q})$. Thus the right hand side of Equation 3.2 becomes $(0, 0, 0, \ldots)$, while the left hand side becomes $\sum_{\text{points}} \partial(\deg_x(f_1, \ldots, f_n) \mod \mathbb{H})$. In summary, we have the following theorem.

Theorem 3.43 (Bézout's theorem over $\mathbb{Q}$)**.** *Let $f_1, \ldots, f_n$ be hypersurfaces in $\mathbb{P}^n_{\mathbb{Q}}$, and let d_i be the degree of f_i for each i. Assume that $f_1, \ldots, f_n$ intersect transversely and that $-n - 1 + \sum_{i=1}^n d_i \equiv 0 \mod 2$. Then we have the following statements.*

(a) We have $\sum_x \operatorname{sign} \deg_x(f_1, \ldots, f_n) = 0$.

(b) We have $\sum_x \partial_2 \deg_x(f_1, \ldots, f_n) \equiv 0 \mod 2$.

(c) For each prime $p \equiv 3 \mod 4$, we have $\sum_x \partial_p \deg_x(f_1, \ldots, f_n) \equiv 0 \mod 4$. (Here, we identify $\partial_p \deg_x(f_1, \ldots, f_n)$ with its image in $\mathbb{Z}/4\mathbb{Z}$.)

(d) For each prime $p \equiv 1 \mod 4$, we have $\sum_x \partial_p \deg_x(f_1, \ldots, f_n) \equiv (0, 0) \mod (2, 2)$. (Here, we identify $\partial_p \deg_x(f_1, \ldots, f_n)$ with its image in $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.)

Remark 3.44. When $x \in U_\ell$ is a rational intersection point of $f_1, \ldots, f_n$, then the local degree $\deg_x(f_1, \ldots, f_n) = \langle J_\ell(x)\rangle$, where $J_\ell(x) \in \mathbb{Q}$ is the signed volume of the parallelpiped spanned by the gradients of $f_1, \ldots, f_n$ at x. Our previous discussion allows us to compute $\partial\langle J_\ell(x)\rangle$, so it remains to consider the case when x is a non-rational intersection point of $f_1, \ldots, f_n$. When $J_\ell(x)$ is a square in the residue field $\mathbb{Q}(x)$ of x, then $\deg_x(f_1, \ldots, f_n) = \mathrm{Tr}_{\mathbb{Q}(x)/\mathbb{Q}}\langle 1\rangle$ is the *trace form* of the field extension $\mathbb{Q}(x)/\mathbb{Q}$. Trace forms of algebraic number fields have been studied extensively. Bayer-Fluckiger and Lenstra [5] showed that if $\mathbb{Q}(x)/\mathbb{Q}$ is an odd degree field extension, then $\mathrm{Tr}_{\mathbb{Q}(x)/\mathbb{Q}}\langle 1\rangle = [\mathbb{Q}(x):\mathbb{Q}]\cdot\langle 1\rangle$. If $\mathbb{Q}(x)/\mathbb{Q}$ is an even degree extension and no Sylow 2-subgroups of $\mathrm{Gal}(\mathbb{Q}(x)/\mathbb{Q})$ are metacyclic, then one can use the Knebusch exact sequence of Witt rings to show that $\mathrm{Tr}_{\mathbb{Q}(x)/\mathbb{Q}}\langle 1\rangle = [\mathbb{Q}(x):\mathbb{Q}]\cdot\langle 1\rangle$ if $\mathbb{Q}(x)$ is totally real and $\mathrm{Tr}_{\mathbb{Q}(x)/\mathbb{Q}}\langle 1\rangle = \frac{[\mathbb{Q}(x):\mathbb{Q}]}{2}\cdot\mathbb{H}$ if $\mathbb{Q}(x)$ is totally imaginary [14]. When $J_\ell(x)$ is not a square, we remark that the discriminant can be computed by

$$\operatorname{disc}\mathrm{Tr}_{\mathbb{Q}(x)/\mathbb{Q}}\langle J_\ell(x)\rangle = \operatorname{norm}(J_\ell(x))\cdot D,$$

where $D = \operatorname{disc}\mathrm{Tr}_{\mathbb{Q}(x)/\mathbb{Q}}\langle 1\rangle$ is the discriminant (up to squares) of the number field $\mathbb{Q}(x)/\mathbb{Q}$.

As an application of Theorem 3.43, we discuss intersections of a line and a conic in $\mathbb{P}^2_{\mathbb{Q}}$.

Example 3.45. Let f be a line and g be a conic in $\mathbb{P}^2_{\mathbb{Q}}$. If f and g intersect with multiplicity 2 at a rational point s, then $\deg_s(f,g) = \mathbb{H}$. If f and g intersect at a non-rational point s, then $i_s(f,g) \geqslant [\mathbb{Q}(s):\mathbb{Q}] > 1$ by Proposition 3.20. Hence $i_s(f,g) = 2$ by the classical version of Bézout's theorem, so s must have a quadratic residue field. Thus f and g intersect transversely at s by Proposition 3.22, and we have $\deg_s(f,g) = \mathrm{Tr}_{\mathbb{Q}(s)/\mathbb{Q}}\langle J(s)\rangle = \mathbb{H}$. This restricts the possible values of $J(s)$. For example, since $\operatorname{disc}\mathbb{H} = -1$, the Hasse–Minkowski principle implies that the discriminant of $\mathrm{Tr}_{\mathbb{Q}(s)/\mathbb{Q}}\langle J(s)\rangle$ must also be equal to -1 up to squares. If D is the field discriminant (up to squares) of $\mathbb{Q}(s)/\mathbb{Q}$, then we have $\mathbb{Q}(s)\cong\mathbb{Q}(\sqrt{D})$. We may thus write $J(s) = a+b\sqrt{D}$, and we have $\operatorname{disc}\mathrm{Tr}_{\mathbb{Q}(s)/\mathbb{Q}}\langle J(s)\rangle = 4D(a^2-b^2D) = D(a^2-b^2D)$ in $\mathbb{Q}^\times/(\mathbb{Q}^\times)^2$. This implies that, up to squares, we have $D(a^2-b^2D)+1 = 0$, so there is a forced relationship between $J(s)$ and the residue field of s. If $\mathbb{Q}(s)\cong\mathbb{Q}(i)$, for example, then we have $a^2+b^2 = 1$ up to squares in $\mathbb{Q}^\times$, so $a^2+b^2 = \operatorname{norm}(J(s))$ must be a square in $\mathbb{Q}^\times$.

Now assume that f and g intersect at two distinct points s,t. By Bézout's theorem, we know that $i_s(f,g) = i_t(f,g) = 1$, so f and g intersect transversely at each of these $\mathbb{Q}$-rational points. Let $J(s)$ (respectively $J(t)$) denote the area of the parallelogram determined by the normal vectors of f and g at s (respectively t). Theorem 3.43 places various restrictions on the possible values of $J(s)$ and $J(t)$. In particular, $J(s)$ and $J(t)$ must have opposite signs and their dyadic valuations must agree mod 2. The local residues of $\langle J(s)\rangle$ and $\langle J(t)\rangle$ at odd primes also constrain the possible intersection types of f and g. For example, it is impossible to have $J(s)$ be a non-square integer (other than -1) and $J(t)$ any integer prime to $J(s)$. Indeed, assume that $J(s) \neq -1$ is a non-square integer and $J(t)$ is an integer prime to $J(s)$, and let p be a prime dividing $J(s)$ such that $v_p(J(s))$ is odd. Since $J(s)$ and $J(t)$ are coprime, we have that $p \nmid J(t)$ and hence $\partial_p\langle J(s)\rangle = \langle a\rangle$ for some $a \in \mathbb{F}_p^\times$ and $\partial_p\langle J(t)\rangle = 0$. Thus $\partial_p\langle J(s)\rangle + \partial_p\langle J(t)\rangle$ is not trivial in $\mathrm{W}(\mathbb{F}_p)$, contradicting Theorem 3.43.

4

The Circles of Apollonius

Given three general circles, there are eight circles that are tangent to all three. This classical theorem, known as the circles of Apollonius, is in fact a corollary of Bézout's theorem. Indeed, the moduli scheme of circles that are tangent to a given circle is a quadric surface in $\mathbb{P}^3$, and the circles of Apollonius correspond to the 2^3 intersection points of three quadric surfaces.

As with many theorems in classical enumerative geometry, Bézout's theorem and the circles of Apollonius require that one work over an algebraically closed field. The $\mathbb{A}^1$-enumerative geometry program applies tools from $\mathbb{A}^1$-homotopy theory to obtain enriched enumerative results over non-algebraically closed fields. In this chapter, we will show that the $\mathbb{A}^1$-enumerative version of the circles of Apollonius is not just a simple corollary of the enriched version of Bézout's theorem. The discrepancy lies in the local information: while the global count is purely hyperbolic, the intersection volume suggested by Bézout's theorem only tells us about the geometry of the quadric surfaces parameterizing tangent circles, rather than the geometry of the tangent circles themselves. The main goal of this chapter is to give the following geometric interpretation of the local indices for the circles of Apollonius:

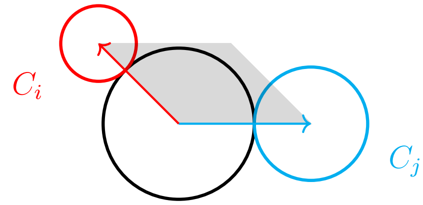

FIGURE 4.1: Geometric interpretation for circles of Apollonius

Lemma 4.1 (See Lemma 4.17)**.** *The local index for a circle C tangent to C_1, C_2, C_3 is given by $\mathrm{Tr}_{k(C)/k}\langle P(C)\rangle$, where $P(C)$ is an alternating sum of the areas of the parallelograms spanned by the center of C and the centers of C_i and C_j for $1 \leqslant i < j \leqslant 3$ (see Figure 4.1).*

4.1 Notation

Throughout this chapter, we let k be a field with $\operatorname{char} k \neq 2$. Let $\mathbb{P}^n_k$ be projective n-space over k. We will be working with circles in the projective plane $\mathbb{P}^2_k$; we denote coordinates on this projective plane by $[x : y : z]$. We will also work with the moduli space of circles in $\mathbb{P}^2_k$, which is isomorphic to $\mathbb{P}^3_k$; we will use the coordinates $[c_0 : c_1 : c_2 : c_3]$ when working with $\mathbb{P}^3_k$. We denote the projective variety cut out by homogeneous polynomials $f_1, \ldots, f_n$ by $\mathbb{V}(f_1, \ldots, f_n)$.

In order to make use of $\mathrm{Tr}_{L/k}$, we will have a running assumption that $k(q)/k$ is a separable extension for any solution $q \in \mathbb{P}^3_k$ to the circles of Apollonius. This separability assumption is guaranteed if k is perfect, if $[k(q) : k] \leqslant 2$ (by our assumption that $\operatorname{char} k \neq 2$), or if $\operatorname{char} k > 8$ (since $[k(q) : k] \leqslant 8$ by the classical version of the circles of Apollonius).

We will frequently write $\operatorname{ind}_p \sigma$ when discussing local indices. The point p corresponds to the image of a tangent circle in the moduli space of circles tangent to a given trio of circles, while σ refers to a section $\sigma : \mathbb{P}^3 \to \mathcal{O}(2)^{\oplus 3}$ determined by this moduli space. The Nisnevich coordinates and local trivializations necessary to make sense of this local index are provided by the author's $\mathbb{A}^1$-enumerative treatment of Bézout's theorem [45].

4.2 Moduli spaces of circles

We begin with a discussion of circles in algebraic geometry, following [26, Section 2.3]. A conic in the projective plane $\mathbb{P}^2_k$ is given by

$$\mathbb{V}(p_0x^2 + p_1xy + p_2xz + p_3y^2 + p_4yz + p_5z^2).$$

The moduli scheme of plane conics is thus isomorphic $\mathbb{P}^5_k$. A circle should be a conic of the form $(x - az)^2 + (y - bz)^2 - r^2z^2 = 0$ for some $a, b, r^2 \in k$. Expanding this out, we have $x^2 + y^2 - 2axz - 2byz + (a^2 + b^2 - r^2)z^2 = 0$. This leads us to the following definition.

Definition 4.2. A *circle* is a conic of the form

$$\mathbb{V}(p_0(x^2 + y^2) + z(p_1x + p_2y + p_3z)).$$

Let $\mathcal{M}_\circ$ be the moduli space of circles in $\mathbb{P}^2_k$. Given $p = [p_0 : p_1 : p_2 : p_3] \in \mathbb{P}^3_k$, let

$$C(p) = \mathbb{V}(p_0(x^2 + y^2) + z(p_1x + p_2y + p_3z)) \in \mathcal{M}_\circ.$$

If $p_0 = 0$, we say that $C([0 : p_1 : p_2 : p_3])$ is a *degenerate circle.*

The definition of C gives us an explicit isomorphism $\mathbb{P}^3_k \cong \mathcal{M}_\circ$.

Proposition 4.3. *Regarded as a map, $C : \mathbb{P}^3_k \to \mathcal{M}_\circ$ is an isomorphism.*

Proof. Note that $C(p)$ does not depend on the choice of representative of p, so $C : \mathbb{P}^3_k \to \mathcal{M}_\circ$ is well-defined. The (well-defined) inverse morphism $C^{-1} : \mathcal{M}_\circ \to \mathbb{P}^3_k$ is given by $C^{-1}\mathbb{V}(p_0(x^2 + y^2) + z(p_1x + p_2y + p_3z)) = [p_0 : p_1 : p_2 : p_3]$. One can readily check that $C \circ C^{-1} = \mathrm{id}_{\mathbb{P}^3_k}$ and $C^{-1} \circ C = \mathrm{id}_{\mathcal{M}_\circ}$. $\square$

Remark 4.4. If $C(p)$ is a non-degenerate circle, then we can solve for the *center* and *radius squared* of $C(p)$ in terms of p. Since $p_0 \neq 0$, we have

$$\begin{aligned} C(p) &= \mathbb{V}(p_0x^2 + p_0y^2 + p_1xz + p_2yz + p_3z^2) \\ &= \mathbb{V}((x + \tfrac{p_1}{2p_0}z)^2 + (y + \tfrac{p_2}{2p_0}z)^2 + (\tfrac{p_3}{p_0} - \tfrac{p_1^2}{4p_0^2} - \tfrac{p_2^2}{4p_0^2})z^2), \end{aligned}$$

which is a circle of radius squared $r^2 := -\frac{p_3}{p_0} + \frac{p_1^2}{4p_0^2} + \frac{p_2^2}{4p_0^2}$ with center $[a : b : 1] := [-\frac{p_1}{2p_0} : -\frac{p_2}{2p_0} : 1]$. We will frequently write

$$\begin{aligned} \tfrac{p_1}{p_0} &= -2a, \\ \tfrac{p_2}{p_0} &= -2b, \\ \tfrac{p_3}{p_0} &= a^2 + b^2 - r^2. \end{aligned}$$

Definition 4.5. The *residue field* or *field of definition* of a circle $C(p) \in \mathcal{M}_\circ$ is the residue field $k(p)$ of the point $p \in \mathbb{P}^3_k$. If $C(p)$ is non-degenerate, then $k(p)/k$ is the minimal field extension such that $a, b, r^2 \in k(p)$. Note in particular that r need not be an element of $k(p)$.

4.2.1 The cone of tangent circles to a given circle

Given a non-degenerate circle $C(p) \in \mathcal{M}_\circ$, we would like to describe the space $Q(p) \subset \mathcal{M}_\circ$ of circles tangent to $C(p)$. By [26, Section 2.3.2], $Q(p)$ is a quadric cone in $\mathcal{M}_\circ$ with cone point $C(p)$. We now describe a directrix for $Q(p)$, which allows us to explicitly solve for $Q(p)$ in terms of p.

Proposition 4.6. *Let $C(p)$ be a non-degenerate circle with radius squared r^2. Any circle of radius squared $(2r)^2$ with center on $C(p)$ is tangent to $C(p)$. (See Figure 4.2.)*

Proof. Let $[a : b : 1]$ be the center of $C(p)$. If $[x_0 : y_0 : 1]$ lies on the circle $C(p)$ (so that $(x_0 - a)^2 + (y_0 - b)^2 - r^2 = 0$), then $C(p)$ is tangent to $S := \mathbb{V}((x - x_0z)^2 + (y -$

$y_0z)^2 - (2r)^2z^2)$ at $q := [2a - x_0 : 2b - y_0 : 1]$. To verify that $q \in C(p)$ and $q \in S$, we simply check

$$(2a - x_0 - a)^2 + (2b - y_0 - b)^2 - r^2 = (2a - 2x_0)^2 + (2b - 2y_0)^2 - (4r)^2 = 0.$$

To verify that $C(p)$ and S are tangent at q, we compute the tangent spaces at q using $T_q\mathbb{V}(f) = \mathbb{V}(\frac{\partial f}{\partial x}|_q \cdot x + \frac{\partial f}{\partial y}|_q \cdot y + \frac{\partial f}{\partial z}|_q \cdot z)$. Thus

$$\begin{aligned} T_qC(p) &= \mathbb{V}(2(a - x_0) \cdot x + 2(b - y_0) \cdot y - 2(a^2 + b^2 + r^2 - ax_0 - by_0) \cdot z), \\ T_qS &= \mathbb{V}(4(a - x_0) \cdot x + 4(b - y_0) \cdot y - 4(2r^2 - x_0^2 - y_0^2 + ax_0 + by_0) \cdot z). \end{aligned}$$

Substituting $2r^2 = r^2 + (x_0 - a)^2 + (y_0 - b)^2$ in the defining equation for T_qS shows that $T_qC(p) = T_qS$ as lines in $\mathbb{P}^2_k$. $\square$

The family of circles of radius squared $(2r)^2$ with center on $C(p)$ will constitute our directrix for $Q(p)$.

Proposition 4.7. *Let $C(p)$ be a non-degenerate circle with center $[a:b:1]$ and radius squared r^2. The family of circles of radius squared $(2r)^2$ with center on $C(p)$ is the circle*

$$\begin{aligned} D := C\mathbb{V}\big(&c_0((a^2 + b^2 + 3r^2)c_0 + ac_1 + bc_2 + c_3), \\ &c_1^2 + c_2^2 + 4c_0((a^2 + b^2 - r^2)c_0 + ac_1 + bc_2)\big). \end{aligned} \tag{4.1}$$

Proof. We obtain the defining equations for the family of circles of radius $(2r)^2$ with center on $C(p)$ by varying $[x_0 : y_0 : 1] \in C(p)$. Parametrically, we have $C(p) = \{[a + r\frac{1-t^2}{1+t^2} : b + r\frac{2t}{1+t^2} : 1] : t \in \mathbb{P}^1\}$. Let E_t be the circle of radius squared $(2r)^2$ with center $[a + r\frac{1-t^2}{1+t^2} : b + r\frac{2t}{1+t^2} : 1]$. Then

$$E_t = C([1 : -2(a + r\tfrac{1-t^2}{1+t^2}) : -2(b + r\tfrac{2t}{1+t^2}) : (a + r\tfrac{1-t^2}{1+t^2})^2 + (b + r\tfrac{2t}{1+t^2})^2 - 4r^2]).$$

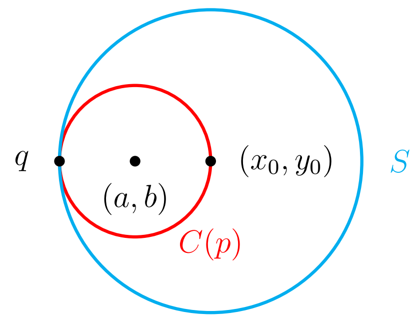

FIGURE 4.2: Circle tangent to $C(p)$

Let $[c_0 : c_1 : c_2 : c_3]$ be coordinates on $\mathbb{P}^3_k$. We then have the implicit description

$$\begin{aligned}\bigcup_{t\in\mathbb{P}^1} E_t &= C\mathbb{V}\big(c_1^2 + c_2^2 - 4c_0c_3 - 16r^2c_0^2,\\ &\qquad (c_1 + 2ac_0)^2 + (c_2 + 2bc_0)^2 - 4r^2c_0^2\big)\\ &= C\mathbb{V}\big(c_1^2 + c_2^2 - 4c_0c_3 - 16r^2c_0^2,\\ &\qquad c_1^2 + c_2^2 + 4c_0((a^2 + b^2 - r^2)c_0 + ac_1 + bc_2)\big).\end{aligned}$$

Substituting $c_1^2 + c_2^2 = -4c_0((a^2 + b^2 - r^2)c_0 + ac_1 + bc_2)$, we find that $\bigcup_{t\in\mathbb{P}^1} E_t$ is given by Equation 4.1. □

Using the vertex p and directrix from Equation 4.1, we now describe the cone $Q(p)$.

Lemma 4.8. *Let $p = [1 : p_1 : p_2 : p_3] \in \mathbb{P}^3_k$. Let $[a : b : 1]$ and r^2 be the center and radius squared, respectively, of $C(p)$. Then*

$$Q(p) = C\mathbb{V}\left((aX + bY + Z)^2 - r^2(X^2 + Y^2)\right),$$

where $X = c_1 - p_1c_0$, $Y = c_2 - p_2c_0$, and $Z = c_3 - p_3c_0$.

Proof. A cone in $\mathbb{P}^3_k$ with vertex $[1:0:0:0]$ is given by the vanishing of $A_1c_1^2 + A_2c_2^2 + A_3c_3^2 + A_4c_1c_3 + A_5c_2c_3 + A_6c_1c_2$ for some $A_1, \ldots, A_6$. In order to translate the vertex to $[1 : p_1 : p_2 : p_3]$, we replace c_1, c_2, and c_3 with X, Y, and Z, respectively. Next, we use the directrix for $Q(p)$ from Proposition 4.7 to solve for $A_1, \ldots, A_6$. We will work in the open affine $\{c_0 \neq 0\} \subset \mathbb{P}^3_k$, after which we will homogenize to obtain the desired equation for $Q(p)$.

On $\{c_0 \neq 0\}$, Equation 4.1 is defined by a circle on the hyperplane $\mathbb{V}((a^2 + b^2 + 3r^2)c_0 + ac_1 + bc_2 + c_3)$. This hyperplane allows us to set $Z|_D = -(a^2 + b^2 + 3r^2 + p_3)c_0 - ac_1 - bc_2$. Remark 4.4 implies that $p_1 = -2a$, $p_2 = -2b$, and $p_3 = a^2 + b^2 - r^2$, so

$$\begin{aligned} Z|_D &= -2(a^2 + b^2 + r^2)c_0 - ac_1 - bc_2 \\ &= -a(c_1 + 2ac_0) - b(c_2 + 2bc_0) - 2r^2 c_0 \\ &= -aX - bY - 2r^2 c_0. \end{aligned}$$

We conclude by expanding $A_1X^2 + A_2Y^2 + A_3Z|_D^2 + A_4XZ|_D + A_5YZ|_D + A_6XY$ and substituting $X = c_1 + 2ac_0$ and $Y = c_2 + 2bc_0$. Comparing to the coefficients of the directrix equation

$$c_1^2 + c_2^2 + 4c_0((a^2 + b^2 - r^2)c_0 + ac_1 + bc_2)$$

allows us to solve for $A_1, \ldots, A_6$. We include some Sage code in Appendix A.1 to perform the algebraic manipulations for us. $\square$

4.2.2 The plane of circles through a point

Given a point $q = [a{:}b{:}1] \in \mathbb{P}^2_k$ away from the line at infinity, we would like to describe the space $V(q) \subset \mathcal{M}_\circ$ of circles through q. In fact, any point in $\mathbb{P}^2_k$ determines an element of $V(q)$, so $V(q)$ is a hyperplane in $\mathcal{M}_\circ$.

Lemma 4.9. *Let $q = [a : b : 1] \in \mathbb{P}^2_k$. Then*

$$V(q) = C\mathbb{V}((a^2 + b^2)c_0 + ac_1 + bc_2 + c_3).$$

Proof. The radius squared of any circle through q is determined by its center. The circle through q with center $[A : B : 1]$ and radius squared r^2 satisfies $(a - A)^2 + (b -$

$B)^2 = r^2$, so this circle is given by

$$\begin{aligned} & C([1 : -2A : -2B : A^2 + B^2 - r^2]) \\ = & C([1 : -2A : -2B : A^2 + B^2 - (a - A)^2 - (b - B)^2]) \\ = & C([1 : -2A : -2B : 2Aa + 2Bb - a^2 - b^2]). \end{aligned}$$

The space of all such circles is defined implicitly by $C\mathbb{V}((a^2+b^2)c_0+ac_1+bc_2+c_3)$. $\square$

Remark 4.10. A point $[a:b:1]$ in $\mathbb{P}^2_k$ can be regarded as a circle with center $[a:b:1]$ and radius squared 0. Under this perspective, the cone $Q([1 : -2a : -2b : a^2 + b^2])$ degenerates to the double plane $C\mathbb{V}((aX + bY + Z)^2)$, which is the plane $V([a:b:1])$ doubled.

4.3 Euler classes and relative orientability

In this section, we compute the fixed count of circles of Apollonius via the Euler class. There are several variants to the circles of Apollonius, because there are two ways in which a circle in $\mathbb{P}^2_k$ can differ from the non-degenerate circles we have considered thus far. First, a degenerate circle (i.e. a circle of the form $C([0:p_1:p_2:p_3])$) is a union of the line $\mathbb{V}(z)$ at infinity with another line in $\mathbb{P}^2_k$. Second, a non-degenerate circle with radius squared 0 is a point. One can thus ask how many circles are tangent to a given set of three objects, where each object may be a circle, line, or point. For simplicity, we will not consider any cases including lines (i.e. degenerate circles).

Each variant of the circles of Apollonius corresponds to studying the intersections of three hypersurfaces, each of the form $Q(p)$ or $V(p)$, in $\mathbb{P}^3_k$. The defining polynomials for $Q(p)$ and $V(p)$ described in Lemmata 4.8 and 4.9 will be used to determine a section $\sigma : \mathbb{P}^3_k \to \mathcal{O}(d_1) \oplus \mathcal{O}(d_2) \oplus \mathcal{O}(d_3)$, where each $d_i = 1$ or 2. Each of these situations is a special case of Bézout's theorem [45]. In this section, we will discuss the Euler class $e(\mathcal{O}(d_1) \oplus \mathcal{O}(d_2) \oplus \mathcal{O}(d_3))$ for each of these cases. In Section 4.4, we will compute the local index $\operatorname{ind}_q \sigma$ [37, Definition 30] of our section at any tangent circle $C(q)$. This local index will give a new invariant on the circles of Apollonius.

4.3.1 CCC

Suppose we are given three general circles $C(p_1), C(p_2), C(p_3) \subset \mathbb{P}^2_k$. The set of circles tangent to these three circles are given by the intersection $Q(p_1) \cap Q(p_2) \cap Q(p_3)$. That is, we are intersecting three degree 2 hypersurfaces in $\mathbb{P}^3_k$. This is a special case of Bézout's theorem, with the defining equations of $Q(p_1), Q(p_2), Q(p_3)$ determining a section of $\mathcal{O}(2)^{\oplus 3} \to \mathbb{P}^3_k$.

Proposition 4.11. *The bundle $\mathcal{O}(2)^{\oplus 3} \to \mathbb{P}^3_k$ is relatively orientable with Euler class $4\mathbb{H}$.*

Proof. The relative orientability is given by [45, Proposition 3.2], and the Euler class is computed in [45, Theorem 4.4]. □

4.3.2 CCP

Suppose we are given two general circles $C(p_1), C(p_2) \subset \mathbb{P}^2_k$ and a point $p_3 \in \mathbb{P}^2_k$. If we consider p_3 as a circle of radius squared 0, then we can again use Proposition 4.11 to check relative orientability and compute the Euler class. However, the local indices in this context fail to be interesting.

Proposition 4.12. *Suppose $C(q_1)$ and $C(q_2)$ are non-degenerate circles with non-zero radius squared, and suppose $C(q_3)$ is a non-degenerate circle with radius squared 0. Suppose $Q(q_i)_{\mathrm{red}}$ intersect transversely at a point q with $k(q)/k$ a separable extension. Then $\operatorname{ind}_q \sigma = \operatorname{Tr}_{k(q)/k} \mathbb{H}$.*

Proof. By [9, Theorem 1.3], we may assume that q is k-rational. Since $C(q_1)$ and $C(q_2)$ are non-degenerate with non-zero radius squared, $Q(q_1)$ and $Q(q_2)$ are reduced.

As discussed in Remark 4.10, $Q(q_3)$ is a double plane. By the transversality assumption on $Q(q_i)_{\mathrm{red}}$, it follows that the intersection multiplicity of the $Q(q_i)$ at q is 2. By [45, Proposition 5.2], it follows that $\operatorname{rank}(\operatorname{ind}_q \sigma) = 2$, so [57, Theorem 2] implies $\operatorname{ind}_q \sigma = \mathbb{H}$. □

The circles of Apollonius for two circles and a point only become interesting when we treat p_3 as a genuine point (rather than as a circle of radius squared 0). Circles tangent to $C(p_1)$ and $C(p_2)$ and through p_3 correspond to the intersection locus $Q(p_1) \cap Q(p_2) \cap V(p_3)$. This is Bézout's theorem for the bundle $\mathcal{O}(2)^{\oplus 2} \oplus \mathcal{O}(1) \to \mathbb{P}^3_k$. However, [45, Proposition 3.2] states that this bundle is not relatively orientable. One can relatively orient the bundle $\mathcal{O}(2)^{\oplus 2} \oplus \mathcal{O}(1)$ relative to the divisor of degenerate circles $\{c_0 = 0\} \subset \mathbb{P}^3_k$ (see [45, Section 3.2]), but the Euler class need not be independent of our choice of section. Nevertheless, we will still discuss the local indices for the cirlce-circle-point problem in Section 4.4.

4.3.3 CPP

Suppose we are given a circle $C(p_1) \subset \mathbb{P}^2_k$ and two general points $p_2, p_3 \in \mathbb{P}^2_k$. If we consider p_2 and p_3 as circles of radius squared 0, then we can again use Proposition 4.11 to check relative orientability and compute the Euler class. However, the local indices in this context are again just hyperbolic forms.

Proposition 4.13. *Suppose $C(q_1)$ is a non-degenerate circle with non-zero radius squared, and suppose $C(q_2)$ and $C(q_3)$ are non-degenerate circles with radius squared 0. Suppose $Q(q_i)_{\mathrm{red}}$ intersect transversely at a point q with $k(q)/k$ a separable extension. Then $\operatorname{ind}_q \sigma = \operatorname{Tr}_{k(q)/k} 2\mathbb{H}$.*

Proof. By [9, Theorem 1.3], we may assume that q is k-rational. Since $C(q_1)$ is non-degenerate with non-zero radius squared, $Q(q_1)$ is reduced. As discussed in Remark 4.10, $Q(q_2)$ and $Q(q_3)$ are double planes. By the transversality assumption on $Q(q_i)_{\mathrm{red}}$, it follows that the intersection multiplicity of the $Q(q_i)$ at q is 4. By [45, Proposition 5.2], it follows that $\operatorname{rank}(\operatorname{ind}_q \sigma) = 4$.

Since q is k-rational, we may change coordinates such that $q = [1:0:0:0]$, the double plane $Q(q_2)$ is defined by $\mathbb{V}(\alpha c_1^2)$ for some $\alpha \in k^\times$, and the double plane $Q(q_3)$ is defined by $\mathbb{V}((\beta c_1 + \gamma c_2)^2)$ for some $\beta \in k$ and $\gamma \in k^\times$. The cone $Q(q_1)$ is defined by $\mathbb{V}(F)$, where $F \in k[c_0, \ldots, c_3]$ is a degree 2 homogeneous polynomial satisfying $F(1,0,0,0) = 0$. Let $f := \frac{1}{c_0^2}F$. Using [36], we calculate $\operatorname{ind}_q \sigma$ by computing the EKL form on the local algebra

$$A := \frac{k[c_1, c_2, c_3]_{(c_1,c_2,c_3)}}{(f, \alpha c_1^2, (\beta c_1 + \gamma c_2)^2)}.$$

The rank of $\operatorname{ind}_q \sigma$ is equal to $\dim_k A$, so any four k-linearly independent elements of A will form a k-basis. Since α and γ are non-zero, it follows that $\{1, c_1, \beta c_1 + \gamma c_2, c_1c_2\}$ is a k-basis of A. Let $E \in A$ be the distinguished socle element [64, (4.7) Korollar], and let $\phi : A \to k$ be any k-linear form satisfying $\phi(E) = 1$. Since $c_1^2 = (\beta c_1 + \gamma c_2)^2 = 0$ and

$$\begin{aligned} c_1c_2(\beta c_1 + \gamma c_2) &= \gamma c_1 c_2^2 \\ &= \gamma^{-1} c_1(-\beta^2 c_1^2 - 2\beta\gamma c_1 c_2) \\ &= 0 \end{aligned}$$

in A, the bilinear form $\Phi : A \times A \to k$ given by $\Phi(a,b) = \phi(ab)$ has the following presentation with respect to the basis $\{1, c_1, \beta c_1 + \gamma c_2, c_1c_2\}$.

	1	c_1	$\beta c_1 + \gamma c_2$	c_1c_2
1	$*$	$*$	$*$	$\phi(c_1c_2)$
c_1	$*$	0	$\gamma \cdot \phi(c_1c_2)$	0
$\beta c_1 + \gamma c_2$	$*$	$\gamma \cdot \phi(c_1c_2)$	0	0
c_1c_2	$\phi(c_1c_2)$	0	0	0

The bilinear form Φ is non-degenerate by [36, Lemma 6], so $\Phi = 2\mathbb{H}$ in $\mathrm{GW}(k)$. □

We will thus treat p_2 and p_3 as genuine points (rather than as circles of radius squared 0). Circles tangent to $C(p_1)$ and through p_2, p_3 correspond to the intersection locus $Q(p_1) \cap V(p_2) \cap V(p_3)$. This is Bézout's theorem for the bundle $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus 2} \to \mathbb{P}^3_k$.

Proposition 4.14. *The bundle* $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus 2} \to \mathbb{P}^3_k$ *is relatively orientable with Euler class* $\mathbb{H}$.

Proof. The relative orientability and Euler class computation can be found in [45, Proposition 3.2 and Theorem 4.4]. □

4.3.4 PPP

Finally, suppose we are given three general points $p_1, p_2, p_3 \in \mathbb{P}^2_k$. If we consider these points as circles of radius squared 0, then Proposition 4.1 again gives us relative orientability and computes the relevant Euler class. However, the intersection of three general double planes in $\mathbb{P}^3_k$ will consist of a single point, so the local index will be equal to the Euler class:

$$\operatorname{ind}_q \sigma = e(\mathcal{O}(2)^{\oplus 3}) = 4\mathbb{H}.$$

As in the previous cases involving points instead of circles, we will treat p_1, p_2, p_3 as genuine points. The unique circle through p_1, p_2, p_3 corresponds to the intersection $V(p_1) \cap V(p_2) \cap V(p_3)$. This is Bézout's theorem for the bundle $\mathcal{O}(1)^{\oplus 3} \to \mathbb{P}^3_k$. As for the circle-circle-point problem, this bundle is not relatively orientable by [45, Proposition 3.2]. One can relatively orient $\mathcal{O}(1)^{\oplus 3}$ relative to the divisor of degenerate circles $\{c_0 = 0\} \subset \mathbb{P}^3_k$, but the Euler class is equal to the local index and will depend on the choice of section.

4.4 Local contributions for Apollonian circles

We now give a geometric interpretation of $\operatorname{ind}_q \sigma$ in terms of the geometry of the relevant circles by analyzing the intersection volume. We will assume that the cones $Q(p_i)$ intersect transversely, which happens whenever the circles $C(p_i)$ are in general position in $\mathbb{P}^2$. (For example, there should not be a single line through the centers of all three circles.)

Given three circles $C(p_i)$ (or points p_i), the intersection volume $\mathrm{Vol}(q)$ at a circle $C(q)$ is defined in terms of the gradients of the cones $Q(p_i)$ (or planes $V(p_i)$) at q. We will assume that $C(p_i)$ and $C(q)$ are non-degenerate circles, so that their c_0 coordinate in $\mathbb{P}^3_k$ is non-zero. This allows us to work in the affine patch $\{c_0 \neq 0\} \subset \mathbb{P}^3_k$, where the twisted covering map of [45, Proposition 3.8] is simply the standard covering map $\{c_0 \neq 0\} \to \mathbb{A}^3_k$. The standard coordinates on $\{c_0 \neq 0\}$ are $(\frac{c_1}{c_0}, \frac{c_2}{c_0}, \frac{c_3}{c_0})$, so the gradient used to calculate $\mathrm{Vol}(q)$ will be $\nabla = (\frac{\partial}{\partial c_1}, \frac{\partial}{\partial c_2}, \frac{\partial}{\partial c_3})$.

Notation 4.15. Let $z_i := [a_i : b_i : 1]$ be the center of $C(p_i)$ (or the point p_i), and let r_i^2 be the radius squared of $C(p_i)$ (or 0 for the point p_i). Similarly, let $\gamma := [\alpha : \beta : 1]$ be the center of the non-degenerate circle $C(q)$. Let ρ^2 be the radius squared of $C(q)$, which is 0 if $C(q)$ is simply the point $[\alpha : \beta : 1]$. Let $\tau_i := [s_i : t_i : 1] \in C(p_i) \cap C(q)$ be the point at which $C(p_i)$ and $C(q)$ are tangent.

We will use the following vectors in $\mathbb{A}^2_k$ (see Figure 4.3):

$$\overrightarrow{\gamma z_i} = (a_i - \alpha, b_i - \beta),$$

$$\overrightarrow{\gamma \tau_i} = (s_i - \alpha, t_i - \beta),$$

$$\overrightarrow{\tau_i z_i} = (a_i - s_i, b_i - t_i).$$

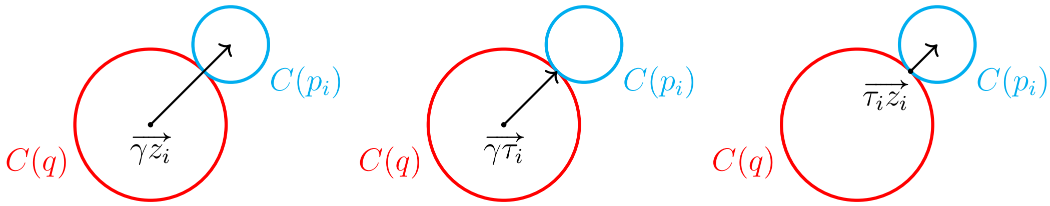

FIGURE 4.3: Externally tangent circles

Finally, define

$$u_i = \begin{cases} \overrightarrow{\tau_i z_i} \cdot \overrightarrow{\gamma z_i} & C(p_i) \text{ a circle,} \\ 1 & p_i \text{ a point} \end{cases}$$

$$v_i = \begin{cases} \overrightarrow{\tau_i z_i} \cdot \overrightarrow{\gamma \tau_i} & C(p_i) \text{ a circle,} \\ 1 & p_i \text{ a point.} \end{cases}$$

Remark 4.16. If k is an ordered field and $r_i^2, \rho^2 > 0$, then we can choose distinguished radii $r_i \in k(\sqrt{r_i^2})$ and $\rho \in k(\sqrt{\rho^2})$ such that $r_i, \rho > 0$. Moreover, any sum of squares is non-negative, so we can define the norm of a vector w to be the non-negative square root of $w \cdot w$. Since the vectors $\overrightarrow{\gamma z_i}$, $\overrightarrow{\gamma \tau_i}$, and $\overrightarrow{\tau z_i}$ are all parallel or anti-parallel, the sign of the dot product of any two of these vectors indicates whether they are parallel or anti-parallel.

In this context, v_i detects whether $C(p_i)$ and $C(q)$ are *externally* tangent (as in Figure 4.3) or *internally* tangent (as in Figures 4.4 and 4.5). Moreover, if $C(p_i)$ and $C(q)$ are internally tangent, then u_i detects whether $\rho > r_i$ (as in Figure 4.4) or $r_i > \rho$ (as in Figure 4.5). In particular:

- $C(p_i)$ and $C(q)$ are externally tangent if and only if $u_i, v_i > 0$.
- $C(p_i)$ and $C(q)$ are internally tangent with $\rho > r_i$ if and only if $u_i < 0$ and $v_i < 0$.
- $C(p_i)$ and $C(q)$ are internally tangent with $r_i > \rho$ if and only if $u_i > 0$ and $v_i < 0$.

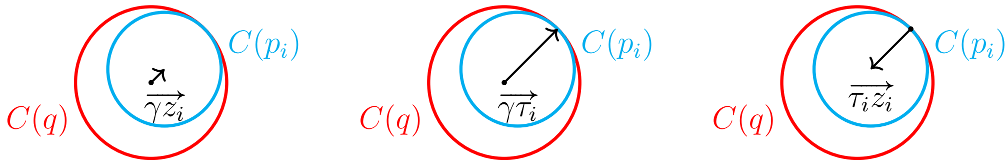

FIGURE 4.4: Internally tangent circles

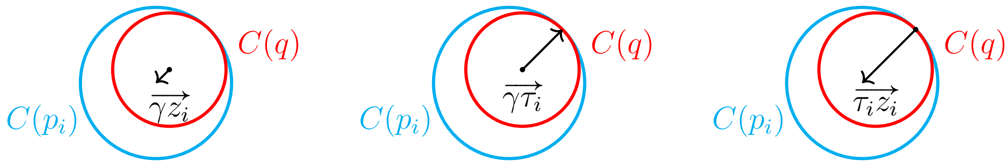

FIGURE 4.5: Internally tangent circles with reversed containment

Lemma 4.17. *If $C(q)$ is tangent to the circles $C(p_i)$ (or points p_i), then the intersection volume is (up to squares)*

$$\mathrm{Vol}(q) = \sum_{\substack{1 \leqslant i \leqslant 3 \\ m<n}} (-1)^{i+1} u_i v_m v_n ((a_m - \alpha)(b_n - \beta) - (a_n - \alpha)(b_m - \beta)).$$

In other words, $\mathrm{Vol}(q)$ is a weighted sum of the signed areas of the parallelograms spanned by $\overrightarrow{\gamma z_m}$ and $\overrightarrow{\gamma z_n}$ (see Figure 4.6), where the weights are given in terms of the dot products u_i, v_m, and v_n.

Proof. If $r_i^2 \neq 0$, we have

$$\begin{aligned} \nabla Q(p_i) = (&2a_i(a_i X + b_i Y + Z) - 2r_i^2 X, \\ &2b_i(a_i X + b_i Y + Z) - 2r_i^2 Y, \\ &2(a_i X + b_i Y + Z)). \end{aligned}$$

Evaluated at q, we have $X = 2(a_i - \alpha)$, $Y = 2(b_i - \beta)$, and $Z = \alpha^2 - a_i^2 + \beta^2 - b_i^2 + r_i^2 - \rho^2$. Thus, evaluated at q, we have

$$\begin{aligned} \nabla Q(p_i)|_q = (&2a_i((a_i - \alpha)^2 + (b_i - \beta)^2 + r_i^2 - \rho^2) - 4r_i^2(a_i - \alpha), \\ &2b_i((a_i - \alpha)^2 + (b_i - \beta)^2 + r_i^2 - \rho^2) - 4r_i^2(b_i - \beta), \\ &2((a_i - \alpha)^2 + (b_i - \beta)^2 + r_i^2 - \rho^2)). \end{aligned}$$

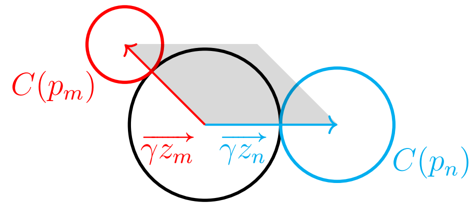

FIGURE 4.6: Parallelogram of tangent circles

If $p_i = [a_i : b_i : 1]$ is a point, then $\nabla V(p_i) = (a_i, b_i, 1)$ is independent of the intersection point q. The intersection volume $\mathrm{Vol}(q)$ is the determinant of the matrix M with rows $\nabla Q(p_i)|_q$ (or $\nabla V(p_i)$). Subtracting α times the third column from the first column of M and β times the third column from the second column of M, we find that $\mathrm{Vol}(q)$ is the determinant of the matrix with i^{th} row

$$\begin{aligned}(2(a_i-\alpha)((a_i-\alpha)^2+(b_i-\beta)^2-r_i^2-\rho^2),\\ 2(b_i-\beta)((a_i-\alpha)^2+(b_i-\beta)^2-r_i^2-\rho^2),\\ 2((a_i-\alpha)^2+(b_i-\beta)^2+r_i^2-\rho^2))\end{aligned} \tag{4.2}$$

if $C(p_i)$ is a circle or

$$(a_i-\alpha, b_i-\beta, 1) = (v_i(a_i-\alpha), v_i(b_i-\beta), u_i)$$

if p_i is a point. Since $[s_i : t_i : 1] \in C(p_i) \cap C(q)$, we have $r_i^2 = (s_i-a_i)^2 + (t_i-b_i)^2$ and $\rho^2 = (s_i-\alpha)^2 + (t_i-\beta)^2$. If $C(p_i)$ is a circle, we may thus substitute for r_i^2 and ρ^2 in Equation 4.2 to obtain $4(v_i(a_i-\alpha), v_i(b_i-\beta), u_i)$. Ignoring the factor of 4 only changes $\mathrm{Vol}(q)$ up to squares, so

$$\mathrm{Vol}(q) = \det\begin{pmatrix} v_1(a_1-\alpha) & v_1(b_1-\beta) & u_1 \\ v_2(a_2-\alpha) & v_2(b_2-\beta) & u_2 \\ v_3(a_3-\alpha) & v_3(b_3-\beta) & u_3 \end{pmatrix}$$

up to squares. $\square$

5

Geometricity

Results in enumerative geometry often consist of equations relating a fixed count of objects to a sum of *local contributions* that depend on the individual objects being counted:

$$\text{fixed count} = \sum_{\text{objects}} \text{local contribution}. \tag{5.1}$$

For example, in the classical version of the circles of Apollonius, the fixed count is 8, and each tangent circle gives a local contribution of 1. In $\mathbb{A}^1$-enumerative geometry, both the fixed count and local contributions are $\mathrm{GW}(k)$-valued rather than integer-valued. In many cases, fixed counts can be computed using a motivic version of the Euler class [2, 37, 41]. Local contributions are computed as a local index, which admits a convenient formula in terms of commutative algebra [9, 12, 36].

In order for Equation 5.1 to be an enumerative *geometric* equation, we need to give geometric descriptions of the local contributions. Giving a meaningful geometric interpretation of the local index, which is a priori an algebraic expression, poses one of the main difficulties in $\mathbb{A}^1$-enumerative geometry.

Question 5.1 (Geometricity)**.** *Are local indices always geometric? Can enumerative problems be classified by the "geometric taxon" of their local indices?*

In a sense to be described in Section 5.1, Bézout's theorem gives a universal geometric interpretation of local contributions. However, this perspective fails to give a satisfactory answer to Question 5.1 — Bézout's theorem gives a geometric interpretation is in terms of the moduli space of the geometric objects in question, rather than in terms of the intrinsic geometry of the objects themselves. We demonstrated this concern in Section 4.4 with the circles of Apollonius as a case study. While the classical statement of the circles of Apollonius can be viewed as a corollary of Bézout's theorem, the geometric interpretation given in Lemma 4.17 shows that the $\mathbb{A}^1$-enumerative situation is more subtle.

Remark 5.2. In light of the previous paragraph, we refine Question 5.1 to ask whether local indices are "intrinsically" geometric, as demonstrated in the following example.

Example 5.3. The second part of Question 5.1 asks for a taxonomy of enumerative problems in terms of the geometric interpretations of their local indices. We propose three potential taxa to give an indication of what this might look like.

(i) *Segre involutions* play a prominent role in Kass–Wickelgren's enriched count of lines on cubic surfaces [37] and Pauli's count of lines on quintic threefolds [55]. The Segre involution associated to a line L on a cubic surface X swaps points $p, q \in L$ such that $T_pX = T_qX$. Kass and Wickelgren show that the local index for lines on cubic surfaces is given by the degree of the Segre involution. The description of Segre involutions associated to lines on quintic threefolds is a little more complicated, but it again relates to swapping points along a line whose tangent spaces coincide. Pauli shows that the local index for lines on quintic threefolds is given by the degree of a product of three Segre involutions. In general, one might hope that the local index for counting lines on a degree $2n-3$ hypersurface in $\mathbb{P}^n$ can be described in terms of Segre involutions.

(ii) There are 2 lines meeting 4 lines in $\mathbb{P}^3$, and in general a finite number of lines meeting $2n-2$ hyperplanes of dimension $n-2$ in $\mathbb{P}^n$. Srinivasan and Wickelgren give an $\mathbb{A}^1$-enumerative account of this story when n is odd [62]. For $n = 3$, the local index is a difference of cross-ratios associated to the geometry of the solution lines L, L', the given lines L_1, L_2, L_3, L_4, their various intersections, and the various planes spanned by pairs of intersecting lines. For larger n, Srinivasan and Wickelgren geometrically interpret the local index by a determinantal formula that again depends on the various intersection points and hyperplanes spanned by pairs of intersecting hyperplanes. This (more complicated) interpretation recovers the difference of cross-ratios when $n = 3$, so this family of enumerative problems share a common geometric local index.

(iii) In [22], the authors give an enriched count of conics meeting 8 lines in $\mathbb{P}^3$. Given a conic C meeting lines $L_1, \ldots, L_8$, the local index is geometrically described in terms of the intersection points $C \cap L_i$ and the slopes of each L_i relative to to the tangent lines $T_{C\cap L_i}C$. The local index comes from an explicit section of the bundle $\mathcal{O}(1)^{\oplus 8} \to \mathbb{P}\mathrm{Sym}^2(\mathcal{S}^\vee)$, where $\mathcal{S} \to \mathbb{G}(2,3)$ is the tautological subbundle on the Grassmannian of 2-planes in $\mathbb{P}^3$.

In general, there are a finite number of degree n plane curves meeting $f(n) := \binom{n+2}{n} + 2$ lines in $\mathbb{P}^3$. The bundle $\mathcal{O}(1)^{\oplus f(n)} \to \mathbb{P}\mathrm{Sym}^n(\mathcal{S}^\vee)$ is relatively orientable if and only if $f(n) = \frac{(n+1)(n+2)}{2} + 2$ is even (see [22, Lemma 3.1]), which happens precisely when n is equivalent to 2 or 3 mod 4. In any case, we again get an explicit section whose associated local index can be described geometrically in terms of the intersection points $C \cap L_i$ and the slopes of L_i relative to

$T_{C\cap L_i}C$ (where C is now a plane curve of degree n).

From this perspective, the enumerative problems of counting plane curves of a given degree meeting lines in $\mathbb{P}^3$ belong to the same geometric taxon. An interesting question is whether the problems of counting $\mathbb{P}^m$-curves of a given degree meeting lines in $\mathbb{P}^n$ (with $m \leqslant n$) also belong to this geometric taxon.

5.1 Intersection volume as a universal local contribution

In classical enumerative geometry, many theorems only become truly *enumerative* when objects are counted with multiplicity. Bézout's theorem is the prototypical example of this phenomenon: unless intersections are counted with multiplicity, the product of degrees merely gives an upper bound to the number of intersections. In this way, intersection multiplicity is a universal local contribution for many classical enumerative problems. Similarly, Bézout's theorem gives a universal geometric interpretation of local contributions in $\mathbb{A}^1$-enumerative geometry. We will first recall this geometric interpretation under a transversality assumption. We will then use Pauli's enrichment of the dynamic degree (see Section 2.1.2 and [55, 56]) to reduce to the transverse case.

Definition 5.4. Let $f_1, \ldots, f_n \in k[x_1, \ldots, x_n]$ with corresponding hypersurfaces $X_i := \mathbb{V}(f_i) \subseteq \mathbb{A}^n_k$. Assume that $p \in \bigcap_i X_i$ is an isolated intersection point with intersection multiplicity $i_p(X_1, \ldots, X_n) = [k(p) : k]$ (that is, X_i intersect transversely at p by [45, Proposition 5.4]). The *intersection volume* $\mathrm{Vol}(p) \in k(p)$ of $f_1, \ldots, f_n$ at p is the volume of the parallelepiped spanned by the gradient vectors $\nabla f_i(p)$. In other terms,

$$\begin{aligned}\mathrm{Vol}(p) &= \det(\nabla f_1(p) \mid \ldots \mid \nabla f_n(p)) \\ &= \mathrm{Jac}(f_1, \ldots, f_n)(p).\end{aligned}$$

In order to compute the intersection volume for a section $\sigma : X \to V$ of a relatively oriented vector bundle, we need Nisnevich coordinates and compatible local trivializations to express σ as an endomorphism of affine space. This intersection volume will not depend on our choices of such data [37, Corollary 31], but we need to show that such data are guaranteed to exist. The existence of Nisnevich coordinates is given by [37, Lemma 19], and the existence of compatible local trivializations was shown in 2.7. We can now show that the local index of a section $\sigma : X \to V$ at a simple zero is always given by an intersection volume.

Proposition 5.5. *Let X be a k-scheme of dimension n. Let $V \to X$ be a relatively orientable vector bundle of rank n, and let $\rho : \det V \otimes \omega_X \xrightarrow{\cong} L^{\otimes 2}$ be a relative orientation of $V \to X$. Let $\sigma : X \to V$ be a section. If $p \in \sigma^{-1}(0)$ is a simple zero with separable residue field $k(p)/k$, then the local index $\operatorname{ind}_p \sigma$ is equal to the intersection volume $\operatorname{Tr}_{k(p)/k}\langle \operatorname{Vol}(p)\rangle$.*

Proof. This is essentially proved in [45, Lemma 5.5], but we repeat the relevant details here. Let $U \subset X$ be an open neighborhood of p with Nisnevich coordinates $\varphi : U \to \mathbb{A}^n_k$, which exist by [37, Lemma 19]. Let $\psi : V|_U \to \mathbb{A}^n_k \times U \to \mathbb{A}^n_k$ be a local trivialization of V compatible with the Nisnevich coordinates φ and the relative orientation (ρ, L), which exists by Proposition 2.7. Then there exist $f_1, \ldots, f_n \in k[x_1, \ldots, x_n]$ such that

$$(f_1, \ldots, f_n) = \psi \circ \sigma \circ \varphi^{-1} : \mathbb{A}^n_k \to \mathbb{A}^n_k.$$

By [37, Corollary 31], the local index $\operatorname{ind}_p \sigma$ is well-defined and independent of our choice of Nisnevich coordinates, compatible trivializations, and functions $f_1, \ldots, f_n$. By [37, Proposition 34] (see also [9]), we have $\operatorname{ind}_p \sigma = \operatorname{Tr}_{k(p)/k} \operatorname{ind}_{\tilde{p}} \sigma_{k(p)}$, where $\tilde{p}$ is the $k(p)$-rational lift of p determined by the extension $k \to k(p)$, and $\sigma_{k(p)}$ is the base change of σ. Since p is a simple zero of σ, we have $\operatorname{ind}_{\tilde{p}} \sigma_{k(p)} = \langle \operatorname{Jac}(f_1, \ldots, f_n)(p)\rangle$ by [36, Proposition 15], which is equal to $\langle \operatorname{Vol}(p)\rangle$ by [45, Section 5.1]. $\square$

Since transversality is a generic condition, Theorem 2.26 implies that we can always interpret the local index $\mathrm{ind}_q \sigma$ as a sum of local indices in the transverse setting, even when q is not a simple zero of σ. By Proposition 5.5, we can always geometrically interpret the local index as a sum of intersection volumes. For example, Theorem 2.26 allows us to remove the transversality assumption in [45, Theorem 1.2]:

Corollary 5.6. *Let $f = (f_1, \ldots, f_n) : \mathbb{A}^n_k \to \mathbb{A}^n_k$ be a morphism with isolated zero p. Let $g_1, \ldots, g_n \in k[\![t]\!][x_1, \ldots, x_n]$ be such that the hypersurfaces $\mathbb{V}(f_i + tg_i) \subseteq \mathbb{P}^n_{k(\!(t)\!)}$ meet transversely. Let $Y = \mathbb{V}(f_1+tg_1, \ldots, f_n+tg_n) \to \operatorname{Spec} k[\![t]\!]$. If $\kappa(z)$ is separable over $k(\!(t)\!)$ for all $z \in Y^p - \{p\}$, then*

$$\deg_p^{\mathbb{A}^1}(f_1, \ldots, f_n) = \sum_{z \in Y^p - \{p\}} \mathrm{Tr}_{\kappa(z)/k(\!(t)\!)} \langle \mathrm{Vol}(z) \rangle,$$

where $\mathrm{Vol}(z)$ is the intersection volume of $f_1 + tg_1, \ldots, f_n + tg_n$ at z.

Proof. We first show that $\kappa(z)/k(\!(t)\!)$ is separable for all $z \in Y^p - \{p\}$. Let $\Phi : Y^p \to \operatorname{Spec} k[\![t]\!]$ be the structure map, which is finite by Proposition 2.19. By [63, Lemma 02GL (1)], our assumption that $k(p)/k$ is separable implies that $p = \operatorname{Spec} k(p)$ is smooth over k. In particular, Φ is smooth at $\Phi^{-1}(0) = (Y^p)_0 = p$. By [63, Lemma 01V9], there exists a non-empty open subset $U \subseteq Y^p$ such that $p \in U$ and $\Phi|_U$ is smooth. But Φ is proper and $Y^p - U$ is closed, so $\Phi(Y^p - U) \subseteq \operatorname{Spec} k[\![t]\!]$ is also closed. Any non-empty closed subset of $\operatorname{Spec} k[\![t]\!]$ contains the sole closed point 0. Since $p \notin Y^p - U$, we have that $\Phi(Y^p - U)$ is empty and hence $Y^p = U$ (as Φ is surjective). It follows that Φ is smooth above the generic point, so $(Y^p)_t \to \operatorname{Spec} k(\!(t)\!)$ is smooth. This map also inherits finiteness from Φ, so $(Y^p)_t \to \operatorname{Spec} k(\!(t)\!)$ is smooth of relative dimension 0 and is therefore étale. It now follows from [63, Lemma 02GL (2)] that $\kappa(z)/k(\!(t)\!)$ is separable for all $z \in (Y^p)_t = Y^p - \{p\}$.

Since we have assumed that $\mathbb{V}(f_i + tg_i)$ meet transversely, [45, Section 3] implies that $\deg_z^{\mathbb{A}^1}(f_1 + tg_1, \ldots, f_n + tg_n) = \mathrm{Tr}_{\kappa(z)/k((t))}\langle \mathrm{Vol}(z)\rangle$. The result now follows from Theorem 2.26. $\square$

As with the dynamic approach, Theorem 2.28 allows us to remove the transversality assumption in [45, Theorem 1.2].

Corollary 5.7. *Assume the conventions of Theorem 2.28. Assume moreover that away from $t = 0$, each fiber $\mathbb{V}(F)_t$ is geometrically reduced. Then for any $c \in k^\times$, the perturbation $\tilde{f} := F|_{t=c} : \mathbb{A}^n_k \to \mathbb{A}^n_k$ of f has a set of zeros $Z \subseteq \tilde{f}^{-1}(0)$ such that*

$$\deg_p^{\mathbb{A}^1}(f) = \sum_{q \in Z} \mathrm{Tr}_{k(q)/k}\langle \mathrm{Vol}(q)\rangle,$$

where $\mathrm{Vol}(q) = \mathrm{Jac}(\tilde{f})(q)$.

Proof. By assumption, $\operatorname{Spec} Q_c$ is geometrically reduced, so the components of $\tilde{f}$ meet transversely at each $q \in Z$. Since F is flat and unramified at $t = c$, we have that $\mathbb{V}(\tilde{f}) \to \operatorname{Spec} k(c) = \operatorname{Spec} k$ is étale [63, Lemma 02GU (2) and (4)]. In particular, $k(q)/k$ is separable for all $q \in Z$ [63, Lemma 02GL (1)]. It follows from [45, Section 5.2] that $\deg_q^{\mathbb{A}^1}(\tilde{f}) = \mathrm{Tr}_{k(q)/k}\langle \mathrm{Vol}(q)\rangle$. The desired result now follows directly from Theorem 2.28. $\square$

In summary, the intersection volume is a universal geometric interpretation of the local indices in $\mathbb{A}^1$-enumerative geometry. However, for most enumerative geometric problems, this interpretation is unsatisfactory — for the circles of Apollonius, the intersection volume at a tangent circle $C(q)$ would tell us about the geometry of the cones $Q(p_i)$ (or planes $V(p_i)$), rather than about the geometry of the circles $C(p_i)$ (or points p_i) and the tangent circle $C(q)$. Question 5.1 asks for a more intrinsic geometric interpretation of $\mathrm{ind}_q \sigma$.

6

Conclusions

We give a brief summary of the results in this dissertation. See Chapter 1 for a more detailed overview of these results. In classical enumerative geometry, Bézout's theorem gives a universal geometric description of the relevant local information — one counts objects with *multiplicity*, and this multiplicity arises from an intersection multiplicity in the moduli space parameterizing the objects being counted. In $\mathbb{A}^1$-enumerative geometry, the local data carries extra structure. One can ask whether this extra structure always encodes geometric information (Question 5.1).

In Chapter 3, we give an arithmetic enrichment of Bézout's theorem. The global count is given by a hyperbolic form (Theorem 3.17), and the local information is given by the volume of the parallelpiped spanned by the gradient directions of the relevant hypersurfaces at a given intersection point (Section 3.4). In Proposition 5.5, we show that this intersection volume, and hence Bézout's theorem, gives a universal geometric description for local indices in $\mathbb{A}^1$-enumerative geometry.

However, the answer given by Bézout's theorem does not satisfactorily solve Question 5.1. This is because the intersection volume is an interpretation in terms of the geometry of the moduli spaces parameterizing the objects being counted, rather than an interpretation in terms of the geometry of the objects themselves. In Chapter 4, we illustrate this point by studying the circles of Apollonius. While the circles of Apollonius in the classical setting are a direct corollary of Bézout's theorem, the intersection volume associated to the circles of Apollonius does not *a priori* tell us anything about the circles at hand. In Lemma 4.17, we give a geometric interpretation for the circles of Apollonius in terms of the parallelograms spanned by the centers of the various circles.

Inspired by Pauli and Wickelgren's dynamic local $\mathbb{A}^1$-degree, we discuss how to compute the local $\mathbb{A}^1$-degree at a given point by working in families (Theorem 2.28). In future work, we will use the familial local $\mathbb{A}^1$-degree to give alternative enriched counts of the circles of Apollonius.

Appendix A

Computations for circles of Apollonius

In this appendix, we include some code and formulas that are useful in computations relevant to Chapter 4.

A.1 Solving for the cone of tangent circles

We include a short piece of Sage code that performs the necessary calculation from Lemma 4.8.

```
var('x','y','z','a','b','r');
var(','.join('c%s'%i for i in range(3)));
var(','.join('A%s'%i for i in range(1,7)));
f = A1*x^2+A2*y^2+A3*z^2+A4*x*z+A5*y*z+A6*x*y;
f = f.subs(z == -a*x-b*y-2*r^2*c0);
f = expand(f.subs(x == c1+2*a*c0, y == c2+2*b*c0));
eqns = [f.coefficient(c0,2) == 4*(a^2+b^2-r^2),\
        f.coefficient(c1,2) == 1,\
        f.coefficient(c2,2) == 1,\
        f.coefficient(c0*c1,1) == 4*a,\
        f.coefficient(c0*c2,1) == 4*b,\
        f.coefficient(c1*c2,1) == 0];
solve(eqns, A1, A2, A3, A4, A5, A6)
```

A.2 Coaklay's equations

For the reader's convenience, we describe Coaklay's solution for the circles of Apollonius [20]. Let k be a field with $\operatorname{char} k \neq 2$. For $1 \leqslant i \leqslant 3$, let $a_i, b_i, r_i^2 \in k$ and $p_i = [1 : -2a_i : -2b_i : a_i^2 + b_i^2 - r_i^2]$, so that $C(p_i) \in \mathcal{M}_\circ$ is the k-rational circle with center $[a_i : b_i : 1]$ and radius squared r_i^2. Let $s = (s_1, s_2, s_3) \in \{1, -1\}^3$. We first define

$$\begin{aligned}\Delta &= \det \begin{pmatrix} a_2 - a_1 & a_3 - a_1 \\ b_2 - b_1 & b_3 - b_1 \end{pmatrix} \\ &= (a_1 - a_2)(b_1 - b_3) - (a_1 - a_3)(b_1 - b_2)\end{aligned}$$

and $D_{ij} = a_i^2 - a_j^2 + b_i^2 - b_j^2 - (r_i^2 - r_j^2)$. Next, let r_i be a square root of r_i^2, and define

$$\begin{aligned}
A_1(s) &= \frac{(s_1r_1 - s_2r_2)(b_1 - b_3) - (s_1r_1 - s_3r_3)(b_1 - b_2)}{\Delta}, \\
B_1(s) &= \frac{(s_1r_1 - s_3r_3)(a_1 - a_2) - (s_1r_1 - s_2r_2)(a_1 - a_3)}{\Delta}, \\
A_2(s) &= \frac{(b_1 - b_3)D_{12} - (b_1 - b_2)D_{13}}{2\Delta}, \\
B_2(s) &= \frac{(a_1 - a_2)D_{13} - (a_1 - a_3)D_{12}}{2\Delta}, \\
M(s) &= A_1(s)s_1r_1 + A_2(s) - a_1, \\
N(s) &= B_1(s)s_1r_1 + B_2(s) - b_1.
\end{aligned}$$

Finally, let

$$\begin{aligned}
f_s(t) = {} & (1 - A_1(s)^2 - B_1(s)^2)(t - s_1r_1)^2 \\
& - 2(M(s)A_1(s) + N(s)B_1(s))(t - s_1r_1) \\
& - M(s)^2 - N(s)^2.
\end{aligned} \tag{A.1}$$

Remark A.1. If k is an ordered field, we can specify that r_i should be non-negative. In general, we cannot consistently choose a "preferred" square root of r_i^2. However, once we have picked r_i, the other square root $-r_i$ will be accounted for by negating s_i in $f_s(t)$.

Remark A.2. Note that $\Delta \neq 0$ if and only if the the three centers $[a_i : b_i : 1]$ are not colinear. It follows that $f_s(t)$ is well-defined if and only if the circles $C(p_i)$ do not have colinear centers.

Theorem A.3 (Coaklay)**.** *The circle $C([1 : -2\alpha_s : -2\beta_s : \alpha_s^2 + \beta_s^2 - \rho_s^2])$ is tangent to $C(p_1), C(p_2), C(p_3)$, where*

$$\alpha_s = A_1(s)\rho_s + A_2(s),$$

$$\beta_s = B_1(s)\rho_s + B_2(s),$$

and ρ_s is a root of $f_s(t)$. Moreover, every circle tangent to $C(p_1), C(p_2), C(p_3)$ is obtained in this manner for some $s \in \{1, -1\}^3$.

Since the roots of f_s and f_{-s} coincide, we can recover all circles of Apollonius with the polynomials $f_{(1,\pm 1,\pm 1)}$.

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Biography

Stephen McKean graduated with a B.S. from the University of Utah in 2017, where he majored in mathematics and minored in physics and German. He was honored with a "top student" award from the German program and the Calvin H. Wilcox memorial scholarship from the Department of Mathematics.

In Fall 2017, Stephen enrolled in the mathematics doctoral program at the Georgia Institute of Technology, where he was awarded the Bob Price travel fellowship and the FESTA fellowship for academics and leadership.

In December 2019, Stephen graduated from Georgia Tech with an M.S. in mathematics in order to transfer to Duke University. While at Duke, he received a Bass instructional fellowship, L.P. and Barbara Smith award, and *Teaching on Purpose* fellowship for his commitment to excellence in teaching.

Stephen has authored or coauthored nine articles [9,10,12,22,45–49], one of which is included in this dissertation [45]. He will continue his academic career as an NSF postdoctoral fellow at Harvard University.