Instructions: Complete 5 of the following 8 problems. Indicate which 5 problems are to be graded by circling their numbers below.

1 2 3 4 5 6 7 8

Problem 1. Let A be a commutative ring and M a finitely-generated A-module. For $m \in M$, let $\text{Ann}(m) = \{a \in A : am = 0\}$.

(a) Prove that for each $m \in M$, $\text{Ann}(m)$ is an ideal of A.

(b) Let $P = \{\text{Ann}(m) : m \in M, m \neq 0\}$. Prove that a maximal element of P is a prime ideal. (An element $\text{Ann}(m)$ is maximal if $\text{Ann}(m) \subseteq \text{Ann}(n)$ implies that $\text{Ann}(m) = \text{Ann}(n)$.)

Problem 2. Let G be a finite group with $x, y \in G$ two elements of order 2. Prove that $\langle x, y \rangle$ is either abelian or isomorphic to a dihedral group.

Problem 3. Let $\alpha, \beta \in \mathbb{C}$ with $\beta \notin \mathbb{Q}$ satisfy $\beta \in \mathbb{Q}(\alpha)$. Prove that $\mathbb{Q}(\alpha)$ is an algebraic extension of $\mathbb{Q}(\beta)$.

Problem 4. Prove or disprove the following statements.

(a) Every UFD is also a PID.

(b) Let R be a UFD. Let $P \subset R$ be a prime ideal with $0 \neq P$ and such that there is no prime ideal strictly between 0 and P. Then P is a principal ideal.

Problem 5. Let V be a finite-dimensional vector space. Show that if a linear operator $T : V \to V$ is nilpotent, then $T^{\dim V} = 0$.

Problem 6. Find $[K : \mathbb{Q}]$, where K is the splitting field of $x^6 - 4x^3 + 1 \in \mathbb{Q}[x]$.

Problem 7. Let $f(x) = x^3 - 3x - 1$.

(a) Show that $f(x)$ is irreducible in $\mathbb{Z}[x]$.

(b) Compute the Galois group of the splitting field of $f(x)$ over $\mathbb{Q}$.

(c) Compute the Galois group of the splitting field of $f(x)$ over $\mathbb{C}$.

Problem 8. Show that there is no simple group of order 616.