Problem 1. Find the Galois group of the splitting field for $f(x) = x^3 - 7$ over $K = \mathbb{Q}(\sqrt{-3})$.

Problem 2. Let ζ be a primitive 37^{th} root of unity, and let $\eta = \zeta + \zeta^{10} + \zeta^{26}$. Determine the Galois group of $\mathbb{Q}(\eta)$ over $\mathbb{Q}$.

Problem 3. Let $f(x) = x^6 + x^3 + 1 = (x^9 - 1)/(x^3 - 1)$.

(a) Prove that $f(x)$ is irreducible over $\mathbb{Q}$.

(b) Find the factorization of $f(x)$ over $\mathbb{F}_{19}$.

Problem 4. Let K be the splitting field over $\mathbb{Q}$ of an irreducible polynomial of degree 3. What are the possibilities for $[K : \mathbb{Q}]$? Give an example to show that each possibility does occur.

Problem 5. Let $f(x)$ be a polynomial of degree n that is irreducible over $\mathbb{Q}$.

(a) If n is prime, prove that the Galois group of $f(x)$ over $\mathbb{Q}$ contains an n cycle.

(b) If n is not prime, show that the Galois group of $f(x)$ over $\mathbb{Q}$ need not contain an n cycle. (Hint: consider the cyclotomic polynomial $\Phi_8(x)$).

Problem 6. Give an example of two field extensions $F/\mathbb{Q}$ and $K/\mathbb{Q}$ with $[F : \mathbb{Q}] = [K : \mathbb{Q}] = 6$ such that $\mathrm{Gal}(F/\mathbb{Q})$ is abelian and $\mathrm{Gal}(K/\mathbb{Q})$ is non-abelian.

Problem 7. Show that for any field F and any integer $d \geq 1$, there exists at most one finite multiplicative subgroup $G \subseteq F^\times$ of order d.

Problem 8. Let $F = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. List all intermediate fields $\mathbb{Q} \subset K \subset F$, and find all elements $\alpha \in F$ such that $F = \mathbb{Q}(\alpha)$.

Problem 9. Let p be a prime and $q = p^n$ for some positive integer n. Show that the map $x \mapsto x^p$ is an automorphism of $\mathbb{F}_q$. Determine all automorphisms of $\mathbb{F}_q$.

Problem 10. Let K/F be a Galois extension with $\mathrm{Gal}(K/F) \cong S_3$. Is it true that K is the splitting field of an irreducible cubic polynomial over F?

Problem 11. Consider the polynomial $f(x) = \dfrac{x^{23}-1}{x-1} = \displaystyle\sum_{i=0}^{22} x^i$. Determine the number of irreducible factors of $f(x)$ over $\mathbb{Q}$, $\mathbb{F}_2$, and $\mathbb{F}_{2048}$.

Problem 12. Find the Galois group of the splitting field of $f(x) = x^3 - x + 1$ over each of the following fields:

(a) $\mathbb{F}_2$

(b) $\mathbb{R}$

(c) $\mathbb{Q}$

Problem 13. Find a factorization of $f(x) = 6x^4 - 4x^3 + 24x^2 - 4x - 8$ into prime elements in $\mathbb{Z}[x]$.

Problem 14. Show that $x^3 - 3x - 1$ is an irreducible element of $\mathbb{Z}[x]$. Compute the Galois group of the splitting field of $f(x)$ over $\mathbb{Q}$ and over $\mathbb{R}$.

Problem 15. Compute the Galois group of $x^4 - x^2 - 6$ over $\mathbb{Q}$.

Problem 16. Let $F \subseteq E$ be an algebraic field extension. Show that $F \subseteq E$ is primitive if and only if the set of intermediate fields $F \subseteq L \subseteq E$ is finite.

Problem 17. Prove that $f(x) = x^4 + 1$ is reducible modulo every prime p but is irreducible in $\mathbb{Q}[x]$.

Problem 18. Using the fact that there are infinitely many primes congruent to 1 modulo m for all $m \in \mathbb{N}$, prove that every finite abelian group appears as the Galois group of some finite Galois extension of $\mathbb{Q}$.

Problem 19. Let K be a finite extension of a field F, and let P be a monic irreducible polynomial in $K[x]$. Prove that there is a non-zero $Q \in K[x]$ such that $PQ \in F[x]$.

Problem 20. Is there an injective field homomorphism from $\mathbb{F}_4 \to \mathbb{F}_{16}$? Is there an injective field homomorphism from $\mathbb{F}_9 \to \mathbb{F}_{27}$? Justify your answer.