Problem 1. Let p, q be prime numbers with $p < q$. Prove that there exists a non-cyclic group of order pq if and only if $p|(q-1)$.

Problem 2. Let G be a finite group, and let H be a subgroup of G of index p, with p prime. If n_H denotes the number of subgroups of G conjugate to H, prove that $n_H = 1$ if H is normal in G, and that $n_H = p$ otherwise.

Problem 3. Let G be a non-abelian finite group with center $Z(G)$. Prove that $\#Z(G) \leq \frac{1}{4}\#G$.

Problem 4. Let G be an abelian group with generators a, b, c and relations

$$\begin{bmatrix} 2 & 10 & 6 \\ 4 & 6 & 12 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0.$$

(a) Find the decomposition of G according to the Fundamental Theorem of finitely-generated abelian groups.

(b) What are cyclic generators corresponding to the components in this decomposition in terms of a, b, c?

Problem 5. Show that all groups of order 35 are cyclic.

Problem 6. Let G be a finite group, and let H be a proper subgroup of G. Prove that the union of all conjugates of H is a proper subset of G. Show that the conclusion need not be true if G is infinite.

Problem 7. Which of the following are class equations for a group G of order 10?

$$1+1+1+2+5 \qquad 1+2+2+5 \qquad 1+2+3+4 \qquad 1+1+2+2+2+2$$

Give an example of a group with each possible class equation, and explain why the rest are impossible.

Problem 8. Let H be the subgroup of S_6 generated by $(1\ 6\ 4\ 25)$ and $(1\ 6)(2\ 5)(3\ 4)$. Let H act on S_6 by conjugation. Show that the set

$$\Sigma = \{(1\ 2)(3\ 5)(4\ 6), (1\ 3)(2\ 4)(5\ 6), (1\ 4)(2\ 5)(3\ 6), (1\ 5)(2\ 6)(3\ 4), (1\ 6)(2\ 3)(4\ 5)\}$$

is invariant under H, thereby defining a homomorphism $\varphi : H \to S_5$. Show that φ is an isomorphism.

Problem 9. Show that every finite group is isomorphic to a subgroup of a simple group.

Problem 10. Let $G = \mathrm{GL}_2(\mathbb{Z}/7\mathbb{Z})$.

(a) Exhibit an element $\sigma \in G$ of order 8.

(b) Describe the structure of a 2-Sylow subgroup of G.

Problem 11. Which of the following groups are isomorphic? Justify your answers.

(a) The multiplicative group of units in $\mathbb{Z}[i]$, with $i^2 = -1$.

(b) $\langle a, b, c \mid a^2 = c^5,\ a^2 = b^4c^4,\ b^2 = c\rangle$

(c) The subgroup of S_4 generated by $(1\ 2)(3\ 4)$ and $(1\ 3)(2\ 4)$.

(d) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/12\mathbb{Z}, \mathbb{Z}/20\mathbb{Z})$

Problem 12. How many isomorphism classes of abelian groups of order 6^4 are there? Explain your answer.

Problem 13. Let G be a finite group and p a prime number. Show that the order of G is not a power of p if and only if G acts transitively on some set X with $|X| \geq 2$ and $(|X|, p) = 1$.

Problem 14. Show that a group of order 80 cannot be simple.

Problem 15. Let G be a group of order 140 and H a subgroup of G of index 4. Show that $H \triangleleft G$.

Problem 16. Let G be a group with subgroup H. If every prime p dividing $|H|$ is at least $[G : H]$, show that H is a normal subgroup of G.

Problem 17. Let p be a prime number and G be a p-group acting on a finite set S. Prove that the number of fixed points of the action is congruent to $|S| \mod p$.

Problem 18. Let G be a group of order p^2q, where p and q are distinct primes. Prove that G has a non-trivial normal subgroup.

Problem 19. For $n \geq 3$, prove that A_n is generated by 3-cycles.

Problem 20. Find five non-abelian groups of order 24 that are pairwise non-isomorphic. Prove that your answer is correct.