Problem 1. Let R be a commutative ring with ideals $I \subset R$ and $J \subset R$. Consider the map $\varphi : I \otimes_R J \to IJ$ given by $a \otimes b \mapsto ab$. Prove or disprove the following statements:

(a) φ is onto.

(b) φ is one-to-one.

Problem 2. Let P be the $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$. Exhibit an exact sequence of $\mathbb{Z}$-modules

$$0 \to M \to N \to N/M \to 0$$

such that

$$0 \to \mathrm{Hom}(N/M, P) \to \mathrm{Hom}(N, P) \to \mathrm{Hom}(M, P) \to 0$$

is not exact.

Problem 3. Let R be a commutative ring with $\sum_{i=0}^{d} c_i x^i = f(x) \in R[x]$ a nilpotent polynomial. Show that each $c_i \in R$ is nilpotent.

Problem 4. Is the ring $\mathbb{Z}[2i]$, where $(2i)^2 = -4$, a principal ideal domain? If not, give an example of a non-principal ideal.

Problem 5. An R-module is called faithful if $rM = 0$ for $r \in R$ implies that $r = 0$. Let M be a finitely generated faithful R-module, and let J be an ideal of R such that $JM = M$. Prove that $J = R$.

Problem 6. Let R be an integral domain. Show that every automorphism of $R[x]$ that is the identity on R is given by $x \mapsto ax + b$, where $a, b \in R$ and a is a unit.

Problem 7. Let R be an integral domain, and let $r \in R$ be a non-zero, non-unit, irreducible element.

(a) If R is a UFD, is $R/(r)$ also a UFD?

(b) If R is a PID, is $R/(r)$ also a PID?

Problem 8. Give an example of a module M over $\mathbb{Z}[x]$ which is torsion-free but not free.

Problem 9. Let $R = \mathbb{Q}[x, y]$. Is R a Euclidean domain? Is R a unique factorization domain?

Problem 10. An R-module M is called irreducible if $M \neq 0$ and the only submodules of M are M and 0. Suppose that R is a commutative ring with M a left R-module. Show that M is irreducible if and only if M is isomorphic to R/I for a maximal ideal I of R.

Problem 11. An algebraic integer is the solution to a monic polynomial with coefficients in $\mathbb{Z}$.

(a) Show that α is an algebraic integer if and only if $\{1, \alpha, \alpha^2, ...\}$ generates a finite-rank $\mathbb{Z}$-module.

(b) Let $f(x) = x^n + a_{n-1}x^{n-1} + ... + a_0 \in \mathbb{Z}[x]$ with root α. Prove that α is a unit in $\mathbb{Z}[\alpha]$ if and only if $a_0 = \pm 1$.

Problem 12. Let R be a commutative ring with a principal maximal ideal M.

(a) Show that there is no ideal I such that $M^2 \subsetneq I \subsetneq M$.

(b) Give an example to show that (a) is false if M is not assumed to be principal.

Problem 13. Suppose that R is a commutative ring such that for every $x \in R$, there is some natural number $n > 1$ such that $x^n = x$. Show that every prime ideal of R is maximal.

Problem 14. Let R be a commutative ring. Let I be an ideal of R such that $xy \in I$ implies either $x \in I$ or $y^n \in I$. Let

$$\sqrt{I} = \{r \in R : r^n \in I \text{ for some } n \in \mathbb{Z}_{>0}\}.$$

Prove that $\sqrt{I}$ is an ideal, is prime, and is the smallest prime ideal containing I.

Problem 15. Let R be a commutative ring with P the set of nilpotent elements in R. Show that P is an ideal, and that R/P has no non-zero nilpotent elements.

Problem 16. Let R and S be commutative rings with $f : R \to S$ a ring homomorphism. Prove that if R is a field, then either f is injective or $S = 0$.

Problem 17. Let R be a principal ideal domain. Show that if $P \neq (0)$ is a prime ideal, then P is maximal.

Problem 18. Let $\varphi : R \to S$ be a ring homomorphism for commutative rings R and S. Suppose that for every prime ideal $I \subset S$, the induced homomorphism $R \to S/I$ is surjective. Prove or disprove that φ is necessarily surjective.

Problem 19. Let R be a commutative ring.

(a) Let I, J be ideals of R, and let P be a prime ideal of R. If $IJ \subset P$, prove that either $I \subset P$ or $J \subset P$.

(b) Let A, B, I be ideals of R. If $I \subset A \cup B$, prove that either $I \subset A$ or $I \subset B$.

Problem 20. Let R be a Noetherian ring. Prove that a surjective homomorphism $\varphi : R \to R$ must be an isomorphism.