Problem 1. Let P be the vector space of all real polynomials and $L : P \to P$ be the linear transformation defined by $L(f) = f + f'$. Prove that L is invertible.

Problem 2. Let $A \in \mathrm{GL}_4(\mathbb{C})$, and suppose A has exactly one eigenvalue λ. Find all possible Jordan forms of A, and prove that $A - \lambda I$ is nilpotent.

Problem 3. Let k be a finite field, and let $M \in \mathrm{GL}_n(k)$. Finally, let $I \in \mathrm{GL}_n(k)$ be the identity matrix. Show that $M^m - I$ is not invertible for some integer $m \geq 1$.

Problem 4. Let $\langle\ ,\ \rangle$ be a positive definite inner product on a finite dimensional real vector space V. Let $S = \{v_1, ..., v_k\}$ be a set of vectors satisfying $\langle v_i, v_j \rangle < 0$ for all $i \neq j$. Prove that $\dim(\mathrm{span}(S)) \geq k - 1$.

Problem 5. Let V be a finite dimensional real vector space, and let $A : V \to V$ be a linear transformation with $A^2 = A$. Show that $\mathrm{trace}(A) = \mathrm{rank}(A)$.

Problem 6. Let V be a finite dimensional vector space over a field k. Show that $V \cong V^*$, where V^* is the vector space of linear transformations $V \to k$.

Problem 7. Let k be a field with $\mathrm{char}(k) \neq 2$, V a finite dimensional vector space over k, and B a symmetric bilinear form on V.

(a) Prove that if $B \neq 0$, then there exists $v \in V$ such that $B(v, v) \neq 0$.

(b) Prove that for any $v \in V$ with $B(v, v) \neq 0$, there exists a subspace $W \subseteq V$ such that $V = Fv \oplus W$ and $W \perp v$.

(c) Prove that there is a basis $\{v_n\}$ of V such that $B(v_i, v_j) = 0$ for all $i \neq j$.

Problem 8. Let V be a finite dimensional vector space and $T : V \to V$ a nonzero linear transformation. Show that if $\ker(T) = \mathrm{im}(T)$, then $\dim(V)$ is an even integer and the minimal polynomial of T is $m(x) = x^2$.

Problem 9. Let k be a field with characteristic p, and let V be a finite dimensional k-vector space. Let $T : V \to V$ be a linear transformation with $T^p = I$.

(a) Show that T has an eigenvector in V.

(b) Show that T is upper-triangular with respect to a suitable basis of V.

Problem 10. Let V and W be finite dimensional vector spaces, and let $T : V \to W$ be a linear transformation. Prove that $\dim(\ker(T)) + \dim(\mathrm{im}(T)) = \dim(V)$.

Problem 11. Find representatives for the (distinct) conjugacy classes of matrices with characteristic polynomial $f(\lambda) = (\lambda^2 + 1)^2$ in

(a) $\mathrm{GL}_4(\mathbb{Q})$.

(b) $\mathrm{GL}_4(\mathbb{C})$.

Problem 12. Let M be an $n \times n$ matrix.

(a) Show that M is invertible if and only if its characteristic polynomial has a non-zero constant term.

(b) Show that if M is invertible, then its inverse M^{-1} may be expressed as a polynomial in M.

Problem 13. If A is an $n \times n$ matrix, then show that $A^n = \alpha_0 I + \alpha_1 A + ... + \alpha_{n-1} A^{n-1}$ for some scalars $\alpha_0, ..., \alpha_{n-1}$.

Problem 14. Let V be a finite dimensional vector space over a field k. Suppose that $A : V \to V$ is a k-linear endomorphism whose minimal polynomial is *not* equal to its characteristic polynomial. Show that there exist k-linear endomorphisms $B, C : V \to V$ with $AB = BA$ and $AC = CA$ but $BC \neq CB$.

Problem 15. Let K be a degree n extension of $\mathbb{Q}$. Let $\sigma_1, ..., \sigma_n : K \hookrightarrow \mathbb{C}$ be the distinct embeddings of K into $\mathbb{C}$, and let $\alpha \in K$. Regarding K as a vector space over $\mathbb{Q}$, let $\varphi : K \to K$ be the linear transformation given by $\varphi(x) = \alpha x$. Show that the eigenvalues of φ are $\sigma_1(\alpha), ..., \sigma_n(\alpha)$.

Problem 16. Let A be a matrix over an algebraically closed field k. Show that $A = A_s + A_n$, where A_s is a diagonalizable matrix, A_n is a nilpotent matrix, and $A_s A_n = A_n A_s$.

Problem 17. Let k be an algebraically closed field, $n \in \mathbb{N}$, and $A \in \mathrm{GL}_n(k)$.

(a) Assume $\mathrm{char}(k) \neq 2$. Show that if A^2 is diagonalizable over k, then A is also diagonalizable over k.

(b) Given an example with $\mathrm{char}(k) = 2$ where A^2 is diagonalizable and A is not diagonalizable.

Problem 18. Without using the fact that they are simultaneously triangularizable, show that two commuting square complex matrices share an eigenvector.

Problem 19. Let F be a field and $n \in \mathbb{N}$.

(a) For $F = \mathbb{R}$, classify up to similarity all matrices $A \in \mathrm{GL}_n(\mathbb{R})$ with $A^3 = A$.

(b) For appropriate F and n, find a matrix $A \in \mathrm{GL}_n(F)$ that is not diagonalizable that satisfies $A^3 = A$.

Problem 20. Let T be a linear operator on a finite dimensional vector space over a field. Prove that $\mathrm{rank}(T^3) + \mathrm{rank}(T) \geq 2 \cdot \mathrm{rank}(T^2)$.