LECTURE 12: PERIOD INTEGRALS AND MODULI OF ELLIPTIC CURVES OVER $\mathbb{C}$

STEPHEN MCKEAN

Last time, we gave two equivalent definitions of elliptic curves over a field k: smooth, projective curves of genus 1 with a k-rational point, and abelian varieties of dimension 1.

Recall that over $\mathbb{C}$, elliptic curves arise as $\mathbb{C}/\Lambda$ for some lattice Λ. To start today's lecture, we'll see that a complex elliptic curve remembers the lattice that it came from.

1. Recovering the lattice from an elliptic curve

Recall that we have a biholomorphism

$$\begin{aligned}\Phi : \mathbb{C}/\Lambda &\to \mathbb{V}(y^2z - 4x^3 + g_2xz^2 + g_3z^3)\\ 0 &\mapsto [0 : 1 : 0]\\ z &\mapsto [\wp(z) : \wp'(z) : 1]\end{aligned}$$

for any non-degenerate lattice Λ (so $g_2^3 - 27g_3^2 \neq 0$). Now suppose I hand you an equation $y^2 = 4x^3 - ax - b$ with $a^3 - 27b^2 \neq 0$. How do I recover Λ?

Well, let E be your complex elliptic curve. Let $\alpha, \beta \subset E$ be closed paths giving a basis of $H_1(E; \mathbb{Z})$. (Remember that E is a torus – try drawing a basis of H_1.) It follows that $\Phi^{-1} \circ \alpha$ and $\Phi^{-1} \circ \beta$ will give a basis of $H_1(\mathbb{C}/\Lambda; \mathbb{Z})$. We have a natural isomorphism

$$\begin{aligned}H_1(\mathbb{C}/\Lambda; \mathbb{Z}) &\to \Lambda\\ \gamma &\mapsto \int_\gamma \mathrm{d}z,\end{aligned}$$

so two generators of Λ can be calculated as $\omega_1 := \int_{\Phi^{-1}\circ\alpha} \mathrm{d}z$ and $\omega_2 := \int_{\Phi^{-1}\circ\beta} \mathrm{d}z$. All that remains is to express these as integrals on E. The chain rule says $\mathrm{d}\wp(z) = \wp'(z)\mathrm{d}z$, so

$$\begin{aligned}\mathrm{d}z &= \frac{\mathrm{d}\wp(z)}{\wp'(z)}\\ &= \Phi^*\left(\frac{\mathrm{d}x}{y}\right).\end{aligned}$$

Now we can compute

$$\begin{aligned}\omega_1 &= \int_{\Phi^{-1}\circ\alpha} \mathrm{d}z \\ &= \int_{\Phi^{-1}\circ\alpha} \Phi^*\left(\frac{\mathrm{d}x}{y}\right) \\ &= \int_\alpha \frac{\mathrm{d}x}{y},\end{aligned}$$

and similarly for ω_2. If you want to write this as an integral involving just one variable, we get

$$\omega_1 = \int_\alpha \frac{\mathrm{d}x}{\sqrt{4x^3 - ax - b}}.$$

Remark 1.1. Elliptic integrals strike again! These elliptic integrals are often called *period* integrals, since they calculate the periods of $\wp$ (which in turn give us the generators of Λ). Inspired by these definitions, you can define a *period* to be any number that you obtain by integrating a differential form over an algebraic variety.

It turns out that essentially every number we know about is a period. Just like algebraic numbers are more complicated than rational numbers but are still accessible to the human mind, so too are periods more complicated than algebraic numbers but still familiar to us.

If you want to learn more about periods, go read the amazing notes of Kontsevich and Zagier [KZ01]. There you will learn that every algebraic number is a period, that periods form a ring that is conjectured (but not known) to not be a field, and that the only examples of known non-periods are bizarre (e.g. [Yos08]). Some well-known numbers like e and

$$\gamma = \int_1^\infty \left(-\frac{1}{x} + \frac{1}{\lfloor x \rfloor}\right) \mathrm{d}x$$

are conjecturally non-periods. It is a straightforward exercise to show that the ring of periods is a countable set (once you have a definition given), and yet this countable set accounts for basically any complex number you can think of.

Exercise 1.2. If I were mean, I might hand you an equation of the form $y^2 = x^3 + ax + b$. This should still define an elliptic curve, but I've changed the variables to obfuscate the connection to $\wp$. Your exercise is to undo my meanness.

Compute the lattice associated to the elliptic curve defined by the vanishing of $y^2 = x^3 + ax + b$. It's okay if you leave your answer in terms of elliptic integrals.

Exercise 1.3. If you actually want to get numbers for ω_1 and ω_2, you better have equations for the paths α and β. Find such equations for the elliptic curve defined by $y^2 = x^3 + ax + b$.

2. Moduli of elliptic curves over $\mathbb{C}$

We have seen that over $\mathbb{C}$, we have a biholomorphism

$$\Phi : \mathbb{C}/\Lambda \to \mathbb{V}(y^2z - 4x^3 + g_2xz^2 + g_3z^3),$$

where $\Phi(0) = [0 : 1 : 0]$ and $\Phi(z) = [\wp(z) : \wp'(z) : 1]$ for $z \neq 0$. Better yet, $\wp$ and $\wp'$ are also functions of Λ, so we get a map

$$\begin{aligned} F : \{\text{lattices in } \mathbb{C}\} &\to \{\text{elliptic curves over } \mathbb{C}\} \\ \Lambda &\mapsto \mathbb{V}(y^2z - 4x^3 + g_2(\Lambda)xz^2 + g_3(\Lambda)z^3). \end{aligned}$$

Using period integration, we also have an inverse map F^{-1}. As written, neither F nor F^{-1} need to be actual functions. We can fix this by modding out by the correct type of "isomorphisms" for lattices and elliptic curves. The real goal is for F to give us a bijection modulo these isomorphisms.

The equivalence relation we need on $\{\text{lattices in } \mathbb{C}\}$ is *homothety.*

Definition 2.1. Two lattices $\Lambda, \Lambda' \subset \mathbb{C}$ are *homothetic* if $\Lambda = c\Lambda'$ for some $c \in \mathbb{C} - \{0\}$.

To see that homothety is necessary, note that changing the differential for period integration yields a homothety of lattices.

Exercise 2.2. Let E be an elliptic curve over $\mathbb{C}$.

(i) Prove that the $\mathbb{C}$-vector space of holomorphic 1-forms on E is 1-dimensional.

(ii) Let ω be a non-trivial holomorphic 1-form on E. Let $\{\alpha, \beta\}$ and $\{\alpha', \beta'\}$ be two bases of the free abelian group $H_2(E; \mathbb{Z})$. Prove that the lattices generated by $\{\int_\alpha \omega, \int_\beta \omega\}$ and $\{\int_{\alpha'} \omega, \int_{\beta'} \omega\}$ are equal.

(iii) Let ω, ω' be two non-trivial holomorphic 1-forms on E. Let $\{\alpha, \beta\}$ be a basis of $H_2(E; \mathbb{Z})$. Prove that the lattices generated by $\{\int_\alpha \omega, \int_\beta \omega\}$ and $\{\int_\alpha \omega', \int_\beta \omega'\}$ are homothetic.

The equivalence relation we need on $\{\text{elliptic curves over } \mathbb{C}\}$ is *isomorphism of* $\mathbb{C}$*-varieties*, which means we need a regular morphism between two elliptic curves whose inverse is also a regular morphism. To give you an idea of what this looks like, let's prove a lemma:

Lemma 2.3. *If Λ, Λ' are homothetic, then $F(\Lambda)$ and $F(\Lambda')$ are isomorphic elliptic curves.*

Proof. Write $\Lambda' = c\Lambda$ for $c \in \mathbb{C} - \{0\}$. Recall that $g_2 = 60G_4$ and $g_3 = 140G_6$. By definition of these Eisenstein series, we have $g_2(c\Lambda) = \frac{1}{c^4} g_2(\Lambda)$ and $g_3(c\Lambda) = \frac{1}{c^6} g_3(\Lambda)$. Write $a = g_2(\Lambda)$ and $b = g_3(\Lambda)$. Then

$$\begin{aligned} F(\Lambda) &= \mathbb{V}(y^2z - 4x^3 + axz^2 + bz^3), \\ F(\Lambda') &= \mathbb{V}(y^2z - 4x^3 + \tfrac{a}{c^4}xz^2 + \tfrac{b}{c^6}z^3). \end{aligned}$$

Now consider the morphism

$$\begin{aligned} T : \mathbb{P}^2 &\to \mathbb{P}^2 \\ [x:y:z] &\mapsto [x:y/c:c^2z]. \end{aligned}$$

This is a regular morphism with inverse

$$T^{-1}([x:y:z]) = [x:cy:z/c^2].$$

Moreover,

$$\begin{aligned} T(F(\Lambda')) &= \mathbb{V}((y/c)^2(c^2z) - 4x^3 + \tfrac{a}{c^4}x(c^2z)^2 + \tfrac{b}{c^6}(c^2z)^3) \\ &= \mathbb{V}(y^2 - 4x^3 + axz^2 + bz^3) \\ &= F(\Lambda). \end{aligned}$$

Similarly, $T^{-1}(F(\Lambda)) = F(\Lambda')$. It follows that $F(\Lambda)$ and $F(\Lambda')$ are isomorphic elliptic curves. □

At this point, we have shown that

$$F : \{\text{lattices in } \mathbb{C}\}/\text{homothety} \to \{\text{elliptic curves over } \mathbb{C}\}/\text{isomorphism}$$

is a bijection onto its image. We now just need to justify why every elliptic curve over $\mathbb{C}$ comes from a lattice. But this boils down to showing that you always get a lattice Λ from the period integrals of an elliptic curve E, and then $\Phi^{-1} : E \to \mathbb{C}\Lambda$ gives us the desired presentation in terms of a lattice.

By rotation and scaling, homothety allows us to assume our lattice is generated by 1 and $\tau \in \mathbb{H}$. So at first glance, it seems like $\mathbb{H}$ is our moduli space of elliptic curves over $\mathbb{C}$. However, there is a hidden symmetry that homothety provides.

Lemma 2.4. *Let Λ_τ and $\Lambda_{\tau'}$ be generated by $\{1, \tau\}$ and $\{1, \tau'\}$, respectively. Then Λ_τ and $\Lambda_{\tau'}$ are homothetic if and only if $\tau' = \gamma \cdot \tau$ for some $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.*

Proof. If Λ_τ and $\Lambda_{\tau'}$ are homothetic, then there exists $\alpha \in \mathbb{C} - \{0\}$ such that $\mathbb{Z}\cdot\tau' + \mathbb{Z} = \mathbb{Z}\cdot\alpha\tau + \mathbb{Z}\cdot\alpha$. In particular, $\tau' = a\alpha\tau + b\alpha$ and $1 = c\alpha\tau + d\alpha$ for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. Thus $\tau' = \frac{a\tau+b}{c\tau+d}$.

Conversely, assume that $\tau' = \frac{a\tau+b}{c\tau+d}$ for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. Then

$$\begin{aligned} (c\tau + d)\Lambda_{\tau'} &= \mathbb{Z}\cdot(a\tau + b) + \mathbb{Z}\cdot(c\tau + d) \\ &= \mathbb{Z}\cdot\tau + \mathbb{Z} \\ &= \Lambda_\tau. \end{aligned}$$

□

Corollary 2.5. *The moduli space of elliptic curves over $\mathbb{C}$ is given by $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$.*

Remark 2.6. As a corollary of our discussion above, every elliptic curve over $\mathbb{C}$ can be written as $\mathbb{V}(y^2z - 4x^3 + axz^2 + bz^3)$ for some $a, b \in \mathbb{C} - \{0\}$. In the following exercise, you'll use a different approach to prove a similar result.

Exercise 2.7. Let E be an elliptic curve over a field k.

(i) Use Riemann–Roch to prove that

$$E = \mathbb{V}(ax^3 + bx^2y + cxy^2 + dy^3 + ex^2z + fxyz + gy^2z + hxz^2 + iyz^2 + jz^3)$$

for some $a, \ldots, j \in k$.

(ii) A *projective change of coordinates* is a function

$$[x : y : z] \mapsto [a_{11}x + a_{21}y + a_{31}z : a_{12}x + a_{22}y + a_{32}z : a_{13}x + a_{23}y + a_{33}z],$$

where $\det(a_{ij}) \neq 0$. Use a projective change of coordinates to show that

$$\mathbb{V}(ax^3 + bx^2y + cxy^2 + dy^3 + ex^2z + fxyz + gy^2z + hxz^2 + iyz^2 + jz^3)$$

is isomorphic to

$$\mathbb{V}(y^2z + mxyz + nyz^2 - x^3 - ox^2z - pxz^2 - qz^3)$$

for some $m, \ldots, q \in k$.

(iii) Show that if $\mathrm{char}(k) \neq 2$, then we can assume that $m = n = 0$.

(iv) Show that if $\mathrm{char}(k) \neq 2$ or 3, then we can assume that either $o = 0$ or $p = 0$.

3. The j-invariant

Recall that $g_2(\Lambda)$ and $g_3(\Lambda)$ are invariants of the lattice Λ, and hence of the elliptic curve $\mathbb{C}/\Lambda$. However, these are not invariant under homothety (and hence under isomorphism), as we saw previously. But if we take an appropriate ratio of polynomials in g_2 and g_3, we can cancel out the effect of scaling and get an invariant of lattices up to homothety (and hence of elliptic curves up to isomorphism). This leads us to the *j-invariant*.

Definition 3.1. The *j-invariant* of a lattice Λ or an elliptic curve $\mathbb{C}/\Lambda$ is the complex number

$$j(\Lambda) = 1728\frac{g_2^3}{g_2^3 - 27g_3^2}.$$

Note that $j(\Lambda)$ is invariant under homothety (or equivalently, under $\mathrm{SL}_2(\mathbb{Z})$ action), so we get a function

$$j(\tau) : \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} \to \mathbb{C}.$$

We won't dive too deep into the amazing world of $j(\tau)$, but this would make a fun semester project if you're interested. For now, here are some important facts:

- $j(\tau) : \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} \to \mathbb{C}$ is a complex analytic isomorphism. In particular, $\mathbb{C}$ is the moduli space of elliptic curves from the perspective of complex analysis.

- The Fourier expansion of $j(\tau)$ in $q = e^{2\pi i\tau}$ is
$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \dots$$
Miraculously, all of these coefficients are *integers*. There's a good reason for this (which we'll talk about next time), but for now I want you to imagine how baffling this must have been when it was first discovered.
- A *modular function* is a meromorphic function $\mathbb{H} \to \mathbb{C}$ that is invariant under $\mathrm{SL}_2(\mathbb{Z})$. It turns out that the set of all modular functions is the field of rational functions $\mathbb{C}(j)$, where j is the j-invariant.

Remark 3.2. We've now seen the moduli space of elliptic curves over $\mathbb{C}$ from a couple perspectives. Next time, we'll try to capture what's really going on with this moduli space, and we'll try to do it in a way that works over any field.

Next time: moduli of elliptic curves and modular forms.

References

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Department of Mathematics, Harvard University

Email address: smckean@math.harvard.edu

URL: shmckean.github.io