LECTURE 14: MODULI STACK OF ELLIPTIC CURVES (CONTINUED)

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Let's wrap up our discussion of elliptic curves.

1. Elliptic curves over a scheme

We're now going to talk about elliptic curves not just over a field, but over any scheme. For a more thorough treatment of this material, see [Ols16, Chapter 13]. Since $\operatorname{Spec}\mathbb{Z}$ is the initial scheme, a moduli space of elliptic curves over $\mathbb{Z}$ would be a best case scenario.

To work at this level of generality, we're going to start by defining this moduli "space" as a category whose objects represent elliptic curves (or rather as a functor out of such a category). In order to give geometric meaning to such a construction, we will need to show that this functor satisfies *descent* with respect to a certain topology (roughly, that we can define this functor on local open sets and then glue these opens together).

Recall the definition of an elliptic curve over a scheme.

Definition 1.1. An *elliptic curve* over a scheme S is a smooth proper morphism $p : E \to S$, equipped with a chosen section $e : S \to E$ such that $(E, e) \times_S \operatorname{Spec}\overline{k(s)}$ is an elliptic curve for each $s \in S$.[1]

A *morphism* of elliptic curves $(p : E \to S, e) \to (p' : E' \to S', e')$ is a pair of scheme morphisms $f : S \to S'$ and $g : E \to E'$ such that $g \circ e = e' \circ f$, and such that the diagram

$$\begin{tikzcd} E \arrow[r, "g"] \arrow[d, "p"] & E' \arrow[d, "p'"] \\ S \arrow[r, "f"] & S' \end{tikzcd}$$

is Cartesian.

Now we can define a category of elliptic curves, which will function as the raw material for our moduli stack.

Definition 1.2. Let $\mathrm{Sch}_{\mathbb{Z}}$ denote the category of $\mathbb{Z}$-schemes. The *moduli stack of elliptic curves* $\mathcal{M}_{1,1}$ is the category over $\mathrm{Sch}_{\mathbb{Z}}$ with:

- objects $(S, (p : E \to S, e))$, where S is a $\mathbb{Z}$-scheme and $(p : E \to S, e)$ is an elliptic curve over S, and
- morphisms of elliptic curves.

[1]This last condition means that the geometric fibers of $p : E \to S$ should all be elliptic curves.

There is a forgetful functor $\pi : \mathcal{M}_{1,1} \to \mathrm{Sch}_{\mathbb{Z}}$ given on objects by $\pi(S, (p : E \to S, e)) = S$ and on morphisms by $\pi(f, g) = f$.

Remark 1.3. The notation $\mathcal{M}_{1,1}$ signifies genus 1 and with 1 base point, as a special case of the very interesting moduli spaces $\mathcal{M}_{g,n}$. Sometimes $\mathcal{M}_{1,1}$ is denoted $\mathcal{M}_{\mathrm{ell}}$.

Remark 1.4. If the category $\mathcal{M}_{1,1}$ had a final object, then every elliptic curve over a scheme could be obtained as a unique pullback of a "universal" elliptic curve over a "universal" base. Automorphisms prevent such a final object from existing: every elliptic curve admits an involution (given by $y \mapsto -y$ if your elliptic curve is presented in Weierstrass form), so there are always at least two morphisms between any pair of elliptic curves. This is further evidence that we need a way to handle these automorphisms if we want the correct notion of moduli space.

We've called $\mathcal{M}_{1,1}$ a stack, but we haven't said what a stack actually is. Very roughly, a stack over $\mathrm{Sch}_{\mathbb{Z}}$ is a sheaf on this category. To make sense of this, we have to talk about (i) what sort of values our sheaf takes and (ii) how to glue with respect to a topology.

For point (i), we need *categories fibered in groupoids.*

Definition 1.5. Let S be a scheme. A category $\mathcal{C}$ with a functor $\pi : \mathcal{C} \to \mathrm{Sch}_S$ is *fibered in groupoids* if:

(a) (Arrow lifitng) For all morphisms $f : U \to V$ in Sch_S and all $y \in \pi^{-1}(V)$, there exists a morphism $\phi : x \to y$ in $\mathcal{C}$ such that $\pi(\phi) = f$.

(b) (Diagram lifting) For all diagrams in $\mathcal{C}$ of the form

$$\pi\left(\begin{array}{ccc} & & x \\ & & \downarrow{\scriptstyle\phi} \\ y & \xrightarrow{\psi} & z \end{array}\right) = \begin{array}{ccc} & & U \\ & & \downarrow{\scriptstyle f} \\ V & \xrightarrow{g} & W, \end{array}$$

and for all $h : U \to V$ factoring f (so $f = g \circ h$), there exists a unique $\chi : x \to y$ in $\mathcal{C}$ such that $\phi = \psi \circ \chi$ and $\pi(\chi) = h$.

Given $U \in \mathrm{Sch}_S$, the *fiber over* U is the category $\mathcal{C}(U)$ whose objects are $\pi^{-1}(U)$ and whose morphisms are $\phi : x \to y$ with $x, y \in \pi^{-1}(U)$ and $\pi(\phi) = \mathrm{id}$.

Exercise 1.6. Here are a few features of this definition that are good to prove:

(i) The morphism $\phi : x \to y$ lifting $f : U \to V$ in part (a) is unique up to unique isomorphism. (Hint: use part (b).)

(ii) A morphism ϕ in $\mathcal{C}$ is an isomorphism if and only if $\pi(\phi)$ is an isomorphism in Sch_S.

(iii) The fiber $\mathcal{C}(U)$ is a *groupoid* (i.e. all morphisms in $\mathcal{C}(U)$ are isomorphisms). This justifies the terminology *fibered in groupoids.*

Proposition 1.7. *The category $\mathcal{M}_{1,1}$ is fibered in groupoids.*

Proof. We have to check arrow lifting and diagram lifting.

(a) Given a map of schemes $f : S \to S'$ and an elliptic curve $E' \in \mathcal{M}_{1,1}(S')$, we need to find a morphism of elliptic curves $\phi : E \to E'$ such that $\pi(\phi) = f$. Since morphisms of elliptic curves are Cartesian squares, we can define $E := E' \times_{S'} S$. Since smoothness and properness are stable under base change, these properties for $E' \to S'$ imply the same properties for $E \to S$. We also have to deal with the base point, but this will simply be the base change of $e' : S' \to E'$.

(b) Suppose we have $S' \to S \leftarrow S''$ given by $\pi(E' \to E \leftarrow E'')$. Given $h : S' \to S''$, we need a unique map of elliptic curves $\chi : E' \to E''$ factoring $E' \to E$ such that $\pi(\chi) = h$. Using h, we can form the pullback $E'' \times_{S''} S'$. Extending this pullback square by $S' \to S \leftarrow S''$ on the bottom and the square $E'' \to E$ over $S'' \to S$, we see that there is a unique map $E' \to E'' \times_{S''} S'$, which in turn admits a unique map $E'' \times_{S''} S' \to E''$. □

So thinking of stacks as sheaves, they should be valued in the categorical analog of groups. Next, we need to discuss a categorical notion of topology.

1.1. **Grothendieck topologies.** When you first learn about topologies, you build them in terms of open subsets around each point in your set. Relativistic math tells you that you should always think of subsets as injective maps, so a topology should be built out of a collection maps satisfying nice properties. This is the philosophy of a Grothendieck topology: we'll put a topology on a category out of collections of certain types of maps over each object.

Definition 1.8. Let $\mathcal{C}$ be a category that admits finite limits. A *Grothendieck topology* on $\mathcal{C}$ is the data of a collection of maps $\{f_i : U_i \to X\}_{i\in I}$, called a *covering*, for each $X \in \mathcal{C}$. Coverings are required to satisfy the following axioms:

(i) If $\{f_i : U_i \to X\}_{i\in I}$ is a covering and $g : Y \to X$ is a morphism in $\mathcal{C}$, then the collection of projection maps $\{U_i \times_X Y \to Y\}_{i\in I}$ is a covering.

(ii) If $\{f_i : U_i \to X\}_{i\in I}$ is a covering and $\{g_j : V_j \to X\}_{j\in J}$ is some collection of maps such that $\{V_j \times_X U_i \to U_i\}_{j\in J}$ is a covering for each i, then $\{g_j : V_j \to X\}_{j\in J}$ is a covering.

(iii) If $\{f_i : U_i \to X\}_{i\in I}$ is a collection of maps such that $f_j : U_j \to X$ admits a section for some $j \in I$, then $\{f_i : U_i \to X\}_{i\in I}$ is a covering.

A *site* is a category with a chosen Grothendieck topology.

This is the sort of definition that takes a little time to sink in. You might want to spend some time thinking about analogies with "open set topology" for each of these axioms.

Exercise 1.9. Let $\mathcal{C}$ be a category with fiber products and a chosen Grothendieck topology. Show that if $\{f_i : U_i \to X\}_{i\in I}$ is a covering and $\{g_j : V_j \to X\}_{j\in J}$ is any collection of maps, then $\{f_i : U_i \to X\}_{i\in I} \cup \{g_j : V_j \to X\}_{j\in J}$ is a covering.

Example 1.10. Here is an important chain of Grothendieck topologies on $\mathrm{Sch}_{\mathbb{Z}}$, listed in increasing order of coarseness.

$$\text{fpqc} \subset \text{fppf} \subset \text{étale} \subset \text{Nisnevich} \subset \text{Zariski}.$$

Here, fpqc means faithfully flat and quasicompact maps, while fppf means faithfully flat maps of finite presentation. Étale maps are the algebro-geometric analog of local homeomorphisms. Nisnevich maps are étale maps that induce isomorphisms on residue fields (so local homeomorphisms that are arithmetically trivial).

It makes sense to talk about sheaves with respect to a given Grothendieck topology. If you're a sheaf in one topology, then you're automatically a sheaf in any coarser topology. Our next goal is to show that $\mathcal{M}_{1,1}$ is a sheaf in the fpqc topology, and hence in all the other topologies listed in the previous example.

We'll continue this story next time.

Next time: Descent, ring spectra, and even periodic cohomology theories

References

[Ols16] Martin Olsson. *Algebraic spaces and stacks*. Vol. 62. American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 2016, pp. xi+298. URL: https://doi.org/10.1090/coll/062.

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