LECTURE 15: DESCENT AND RING SPECTRA

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Last time, we saw that $\pi : \mathcal{M}_{1,1} \to \mathrm{Sch}_{\mathbb{Z}}$ gives $\mathcal{M}_{1,1}$ the structure of a category fibered in groupoids. Today, we'll finish up the story of $\mathcal{M}_{1,1}$ being a stack.

1. Descent

Recall that a *Grothendieck topology* on a category is a collection of morphisms for each object. These collections are called *coverings*, and they satisfy certain axioms that are meant to mimic the axioms used to defined topologies on sets in terms of open subsets. Here's an exercise to help you get used to this definition.

Exercise 1.1. Let $\mathcal{C}$ be a category with fiber products and a chosen Grothendieck topology. Show that if $\{f_i : U_i \to X\}_{i\in I}$ is a covering and $\{g_j : V_j \to X\}_{j\in J}$ is any collection of maps, then $\{f_i : U_i \to X\}_{i\in I} \cup \{g_j : V_j \to X\}_{j\in J}$ is a covering.

Example 1.2. Here is an important chain of Grothendieck topologies on $\mathrm{Sch}_{\mathbb{Z}}$, listed in increasing order of coarseness.

$$\text{fpqc} \subset \text{fppf} \subset \text{étale} \subset \text{Nisnevich} \subset \text{Zariski}.$$

Here, fpqc means faithfully flat and quasicompact maps (with an oft forgotten finiteness condition), while fppf means faithfully flat maps of finite presentation. Étale maps are the algebro-geometric analog of local homeomorphisms. Nisnevich maps are étale maps that induce isomorphisms on residue fields (so local homeomorphisms that are arithmetically trivial).

It makes sense to talk about sheaves with respect to a given Grothendieck topology.

Definition 1.3. A *stack* over a scheme S (with respect to a Grothendieck topology τ on Sch_S) is a category $\mathcal{C}$ fibered in groupoids over Sch_S such that

$$\begin{aligned} \mathrm{Sch}_S &\to \mathrm{Sets} \\ U &\mapsto \mathcal{C}(U) \end{aligned}$$

is a *sheaf of groupoids.* This consists of two conditions:

(i) (Objects glue) For all coverings $\{U_i \to U\}$, all $x_i \in \mathcal{C}(U_i)$, and all (iso)morphisms $\alpha_{ij} : x_i|_{U_i\times_U U_j} \to x_j|_{U_i\times_U U_j}$ satisfying the cocycle condition $\alpha_{ik} = \alpha_{jk} \circ \alpha_{ij}$ over $U_i \times_U U_j \times_U U_k$, there exists an object $x \in \mathcal{C}(U)$ (unique up to isomorphism) and (iso)morphisms $\alpha_i : x|_{U_i} \to x_i$ in $\mathcal{C}(U_i)$ such that $\alpha_{ij} = \alpha_j|_{U_i\times_U U_j} \circ (\alpha_i|_{U_i\times_U U_j})^{-1}$.

(ii) (Morphisms glue) Given a covering $\{U_i \to U\}$, objects $x, y \in \mathcal{C}(U)$, and (iso)morphisms $\alpha_i : x|_{U_i} \to y|_{U_i}$ such that $\alpha_i|_{U_i \times_U U_j} = \alpha_j|_{U_i \times_U U_j}$, there is a unique (iso)morphism $\alpha : x \to y$ such that $\alpha|_{U_i} = \alpha_i$.

Remark 1.4. Another common name for condition (i) is: "the descent datum is effective." A category fibered in groupoids that satisfies morphism gluing is sometimes called a *prestack*.

This definition might seem like a lot at first, but that's exactly how sheaves felt the first time you learned them.[1] Here's an exercise to get you comfortable with this definition.

Exercise 1.5. Let Bun_r/S be the category of rank r vector bundles over S-schemes. Objects of this category are rank r vector bundles $E \to X$, where X is an S-scheme. Morphisms in this category are pullbacks (i.e. there is a morphism $E' \to E$ if and only if there is a (necessarily unique) S-scheme morphism $\phi : X' \to X$ such that $E' \cong \phi^* E$.

(i) Define a forgetful functor $\pi : \mathrm{Bun}_r/S \to \mathrm{Sch}_S$.

(ii) Prove that π gives Bun_r/S the structure of a category fibered in groupoids.

(iii) Prove that Bun_r/S is a stack in the Zariski topology.

Note that for gluing objects, it is not true that a vector bundle $E \to X$ can be recovered from its restrictions $E|_{U_i}$, where $\{U_i \to X\}$ is a covering. This is because any vector bundle is trivialized by some covering. So for this part, you also need to use the isomorphisms $\alpha_{ij} : E_i|_{U_i \times_X U_j} \to E_j|_{U_i \times_X U_j}$ satisfying the cocycle condition.

Gluing morphisms is easier: you just need to show that an isomorphism between two vector bundles $E \to X$ and $E' \to X$ can be defined locally on a covering and glued in a unique way if the isomorphisms $E|_{U_i} \to E'|_{U_i}$ agree on overlaps.

If you're a sheaf in one topology, then you're automatically a sheaf in any coarser topology. It turns out that $\mathcal{M}_{1,1}$ is an fpqc stack, and hence a stack in the fppf, étale, Nisnevich, and Zariski topologies as well. Because we're behind in the class and the proof of even Zariski descent is a bit lengthy, I'll punt this one to some nice notes (written by homotopy theorists).

Theorem 1.6. *$\mathcal{M}_{1,1}$ is an fpqc stack.*

Proof. See Section 3 of `https://webspace.science.uu.nl/~meier007/Mell.pdf`. □

There's much more to the story that we'll sadly have to skip. If you're interested in learning more, you can go read about representability of stacks, stacks with certain representability properties (like Deligne–Mumford and Artin stacks), and much more in Vistoli's notes: `https://arxiv.org/pdf/math/0412512.pdf`.

[1] Or at least that's how I personally felt.

2. Ring spectra

We're now taking a hard pivot back to homotopy theory. The moduli stack of elliptic curves will return in the final episodes of the tmf cinematic universe.

Recall that a spectrum is, roughly, a sequence of topological spaces, together with maps connecting sequential spaces in a precise way.

Definition 2.1. A *spectrum* is a sequence $(X_n)_{n\in\mathbb{N}}$ of pointed topological spaces, together with base point preserving continuous maps $\sigma_n : S^1 \wedge X_n \to X_{n+1}$ for all $n \in \mathbb{N}$.

A *morphism* of spectra is a sequence of continuous maps $X_n \to Y_n$ that fit into commutative squares with the structure maps.

Genera are ring homomorphisms of the form $\Omega_*^G \to R$, where Ω_*^G is the G-cobordism ring for some Lie group G. We saw in the case of $G = \mathrm{SO}$ that there is a Thom spectrum MSO such that $\pi_*\mathrm{MSO} \cong \Omega_*^{\mathrm{SO}}$. Likewise, one can define the Thom spectrum MG for any Lie group G and prove that $\pi_* MG \cong \Omega_*^G$.

This gives rise to a natural question: is there a spectrum S such that $\pi_* S \cong R$, and a map of spectra $MG \to S$ such that $\pi_*(MG \to S)$ recovers the genus? Lifting genera to the level of spectra is an interesting question, because the extra structure (e.g. higher homotopy groups) can tell us new things about the genus. Topological modular forms were first discovered by answering this sort of question.

In order for any of this to make sense, notice that $\pi_* S$ needs to be a *ring*. Earlier in the class, we wondered about how $\pi_*\mathrm{MSO}$ could recover the graded ring structure of Ω_*^{SO}, instead of just the sequence of abelian groups $\Omega_0^{\mathrm{SO}}, \Omega_1^{\mathrm{SO}}, \ldots$ This is a special feature of *ring spectra*.

In order to define ring spectra, we need a suitable product operation for spectra. This is given by the *smash product*, which is induced from the smash product of pointed topological spaces. In contrast to the smash product of spaces, which is fairly straightforward, smash products of spectra are a real pain to get right. There are lots of models for defining the smash product, but it will be easiest for us to talk about Adams's old-fashioned approach.

Definition 2.2. Let $X = (X_n, \sigma_n)$ and $Y = (Y_n, \tau_n)$ be spectra. The *smash product* $X \wedge Y$ has spaces

$$(X \wedge Y)_{2n} := X_n \wedge Y_n,$$
$$(X \wedge Y)_{2n+1} := S^1 \wedge X_n \wedge Y_n$$

and structure maps

$$\rho_{2n} := \mathrm{id} : S^1 \wedge X_n \wedge Y_n \to S^1 \wedge X_n \wedge Y_n,$$
$$\rho_{2n+1} := \sigma_n \wedge \tau_n : S^1 \wedge S^1 \wedge X_n \wedge Y_n \simeq (S^1 \wedge X_n) \wedge (S^{1\wedge}_1 Y_n) \to X_{n+1} \wedge Y_{n+1}.$$

While this definition is simple enough, Adams's smash product is only well-behaved if $S^1 \wedge -$ is invertible. But once we stabilize, $\wedge$ gives the category of spectra the structure of a symmetric monoidal category. The monoidal unit is given by the sphere spectrum $\mathbb{S} := (S^0, S^1, S^2, \ldots)$.

Definition 2.3. A map of spectra $f : X \to Y$ is a *weak equivalence* if the induced map $f_* : \pi_n X \to \pi_n Y$ is an isomorphism for all n. The *stable homotopy category*, denoted SH, is the category of spectra up to weak equivalences. If two spectra X and Y are weakly equivalent, we write $X \simeq Y$.

The following exercise is to verify that $(\mathrm{SH}, \wedge)$ is symmetric monoidal with unit $\mathbb{S}$.

Exercise 2.4. Show that $X \wedge \mathbb{S} \simeq X$ for any spectrum X. Also show that $X \wedge Y \simeq Y \wedge X$ for any spectra X and Y.

Now we can define ring spectra.

Definition 2.5. A *ring spectrum* is a spectrum E, together with a multiplication map $\mu : E \wedge E \to E$, a unit map $u : \mathbb{S} \to E$, and homotopies

$$\mu(\mathrm{id} \wedge \mu) \sim \mu(\mu \wedge \mathrm{id}),$$
$$\mu(\mathrm{id} \wedge u) \sim \mathrm{id} \sim \mu(u \wedge \mathrm{id}),$$

which witness associativity and unitality, respectively.

Example 2.6. The sphere spectrum is a ring spectrum (which you should check). It turns out that MSO is also a ring spectrum, but not every Thom spectrum is a ring spectrum! If you think back to our discussion of $\pi_*\mathrm{MSO} \otimes \mathbb{Q}$, it might not be so surprising that if G is an H-space, then MG is a ring spectrum. If you're looking for a homotopy-flavored semester paper, writing about which Thom spectra are ring spectra would be a fun dive into some classical literature.

We'll close with an exercise:

Exercise 2.7. Let E be a ring spectrum. Prove that the ring structure on E induces a graded ring structure on $\pi_* E$.

If you're interested in learning about homotopy theory, I strongly encourage you to think about this exercise. If you're really motivated, try to use your proof to write down explicitly what the ring structure on $\pi_*\mathbb{S}$ might look like. If you're less excited about homotopy theory, then you can take this exercise as a fact and notice that this gives us the desired ring structure on $\pi_*\mathbb{S}$, $\pi_*\mathrm{MSO}$, and so on.

Next time, we'll discuss a special class of spectra and their associated cohomology theories.

Next time: even periodic cohomology theories, formal group laws, and elliptic cohomology.

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