LECTURE 16: RING SPECTRA, EVEN PERIODIC COHOMOLOGY, AND COMPLEX ORIENTATIONS

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1. Ring spectra

Recall that we want to lift genera, which are ring homomorphisms $\Omega_*^G \to R$, to the level of spectra. In order to do this, we need the notion of a ring spectrum.

Definition 1.1. A *ring spectrum* is a spectrum E, together with a multiplication map $\mu : E \wedge E \to E$, a unit map $u : \mathbb{S} \to E$, and homotopies

$$\mu(\mathrm{id} \wedge \mu) \sim \mu(\mu \wedge \mathrm{id}),$$
$$\mu(\mathrm{id} \wedge u) \sim \mathrm{id} \sim \mu(u \wedge \mathrm{id}),$$

which witness associativity and unitality, respectively.

Remark 1.2. There was a complaint last time about what $\sim$ is supposed to mean in the previous definition. The glib answer is that we're requiring the diagrams

$$\begin{array}{ccc} E \wedge E \wedge E & \xrightarrow{\mu \wedge \mathrm{id}} & E \wedge E \\ \downarrow{\scriptstyle \mathrm{id}\wedge\mu} & & \downarrow{\scriptstyle \mu} \\ E \wedge E & \xrightarrow{\mu} & E \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} \mathbb{S} \wedge E & \xrightarrow{\cong} & E & \xleftarrow{\cong} & E \wedge \mathbb{S} \\ \downarrow{\scriptstyle u\wedge\mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}\wedge u} \\ E \wedge E & \xrightarrow{\mu} & E & \xleftarrow{\mu} & E \wedge E \end{array}$$

to commute in the stable homotopy category. But this still doesn't tell you what a homotopy of maps $f, g : X \to Y$ of spectra should be. Well, a homotopy of maps of spectra should be a map of spectra $h : X \wedge I_+ \to Y$, where $(X \wedge I_+)_n = X_n \wedge ([0,1] \cup \{*\})$, such that $h_0 = f$ and $h_1 = g$.

For this definition to truly work, we need to be a little more flexible in what we mean by a map of spectra. A map of spectra $X \to Y$ does not actually need to be defined for all $X_n \to Y_n$, but rather needs to be defined on all but finitely many $X_n \to Y_n$. Another way to think of this is that $X \to Y$ needs to be defined on all $X_n \to Y_n$ for all $n \geq N$ for some N.

This is all related to the subtleties that go into defining the stable homotopy category. Another good semester project would be to write about two or three different ways of defining SH and proving that you get equivalent categories.

Example 1.3. The sphere spectrum is a ring spectrum (which you should check). It turns out that MSO is also a ring spectrum, but not every Thom spectrum is a ring spectrum! If you think back to our discussion of $\pi_*\mathrm{MSO} \otimes \mathbb{Q}$, it might not be so surprising that if G is an H-space, then MG is a ring spectrum. If you're looking for a homotopy-flavored semester paper, writing about which Thom spectra are ring spectra would be a fun dive into some classical literature.

Example 1.4. Let $\mathrm{U}(n)$ denote the degree n unitary group. Unitary bundles are classified by maps into the classifying space $\mathrm{BU}(n)$, where you pull back the universal bundle $\xi_n \to \mathrm{BU}(n)$ to get your unitary bundle. The direct sum of unitary bundles is classified by a map $\mathrm{BU}(n) \times \mathrm{BU}(m) \to \mathrm{BU}(n+m)$, fitting into a pullback square

$$\begin{array}{ccc} \xi_n \oplus \xi_m & \longrightarrow & \xi_{n+m} \\ \downarrow & & \downarrow \\ \mathrm{BU}(n) \times \mathrm{BU}(m) & \longrightarrow & \mathrm{BU}(n+m). \end{array}$$

Since $\mathrm{Th}(V \oplus W) \simeq \mathrm{Th}(V) \wedge \mathrm{Th}(W)$, this diagram gives us a map $\mathrm{Th}(\xi_n) \wedge \mathrm{Th}(\xi_m) \to \mathrm{Th}(\xi_{n+m})$. The Thom spectrum MU associated to BU has spaces $\mathrm{Th}(\xi_n)$, so we have just constructed a map $\mathrm{MU}(n) \wedge \mathrm{MU}(m) \to \mathrm{MU}(n+m)$. Taking colimits as n and m go to ∞ gives us a map of spectra $\mu : \mathrm{MU} \wedge \mathrm{MU} \to \mathrm{MU}$. It turns out that this multiplication is part of the ring structure on MU.

The ring structure on a spectrum gives you a graded ring structure on its homotopy groups:

Exercise 1.5. Let E be a ring spectrum. Prove that the ring structure on E induces a graded ring structure on $\pi_* E$.

If you're interested in learning about homotopy theory, I strongly encourage you to think about this exercise. If you're really motivated, try to use your proof to write down explicitly what the ring structure on $\pi_*\mathbb{S}$ might look like. If you're less excited about homotopy theory, then you can take this exercise as a fact and notice that this gives us the desired ring structure on $\pi_*\mathbb{S}$, $\pi_*\mathrm{MSO}$, and so on.

2. Even periodic cohomology theories

We're now going to talk about an important class of spectra (and their associated cohomology theories).

Definition 2.1. If $E = (E_n, \sigma_n)$ is a spectrum and X is a pointed topological space, then the *E-valued cohomology* of X is given by

$$E^n(X) := [\Sigma^\infty X, \Sigma^n E].$$

Here, $\Sigma^\infty X := (X, \Sigma X, \Sigma^2 X, \ldots)$ is the *suspension spectrum* associated to X, and $\Sigma^n E = (\ldots, \Sigma^n E_0, \Sigma^n E_1, \ldots)$ is the n-fold suspension of E. When E is an *Ω-spectrum* (meaning that $E_n \simeq \Omega E_{n+1}$ for all n), we get a simpler formula

$$E^n(X) = [X, E_n].$$

In both of these formulas, $[-,-]$ denotes the set of homotopy classes of maps (whether maps of spectra or maps of pointed spaces). When E is an Ω-spectrum, the spaces E_n are H-spaces, which yields an abelian group structure on $[-, E_n]$.

Example 2.2. K-theory is a cohomology theory where $K^0(X)$ is the group of virtual complex vector bundles on X. Complex vector bundles are classified by BU, and virtual complex bundles are classified by $\mathrm{BU}\times\mathbb{Z}$ (the $\mathbb{Z}$ records the virtual dimension). We can build an Ω-spectrum

$$\mathrm{KU} := (\mathrm{BU}\times\mathbb{Z}, \Omega(\mathrm{BU}\times\mathbb{Z}), \Omega^2(\mathrm{BU}\times\mathbb{Z}), \dots),$$

which at first looks unwieldly. However, one version of Bott periodicity gives a homotopy equivalence $\Omega^2(\mathrm{BU}\times\mathbb{Z}) \simeq \mathrm{BU}\times\mathbb{Z}$. In particular, $\mathrm{KU}_{n+2}\simeq \mathrm{KU}_n$ for all n. This implies that $\mathrm{KU}^{n+2}(X)\cong \mathrm{KU}^n(X)$ for all n and any pointed space X, so complex K-theory is a *periodic* cohomology theory. One can even show that this periodicity is given by multiplication by the *Bott class*, which is an invertible element $\beta\in\mathrm{KU}^*(\mathrm{pt})$ of degree -2.

Since $\Omega(\mathrm{BU}\times\mathbb{Z})\simeq \mathrm{U}$, we can calculate

$$\mathrm{KU}^{2n}(\mathrm{pt}) \cong \pi_0(\mathrm{BU}\times\mathbb{Z}) = \mathbb{Z},$$
$$\mathrm{KU}^{2n+1}(\mathrm{pt}) \cong \pi_0(\mathrm{U}) = 0.$$

That is, the cohomology $\mathrm{KU}^*(\mathrm{pt})$ is concentrated in even degrees, so complex K-theory is an *even* cohomology theory. Together, these properties inspire a definition.

Definition 2.3. A spectrum (or cohomology theory) E is *even periodic* if (a) $E^i(\mathrm{pt}) = 0$ for all odd i and (b) there exists $\beta\in E^{-2}(\mathrm{pt})$ such that β is invertible in $E^*(\mathrm{pt})$.

Example 2.4. The spectrum MU is even, because $\pi_*\mathrm{MU}\cong\mathbb{Z}[x_1, x_2, \dots]$ with $|x_i| = -2i$, but this also shows us that MU is not periodic. You can build an even periodic spectrum out of MU by taking

$$\mathrm{MP} := \bigvee_{k\in\mathbb{Z}} (\mathbb{S}^{2k}\wedge \mathrm{MU}).$$

Here, the wedge sum of two spectra is built by taking wedge sums of their constituent spaces, which is well-defined since $S^1\wedge(\bigvee_i E_n^i) = \bigvee_i (S^1\wedge E_n^i)$. (Categorically speaking, the stable homotopy category has coproducts.)

At the level of cohomology, this gives us

$$\mathrm{MP}^n(X) = \prod_{k\in\mathbb{Z}} \mathrm{MU}^{n+2k}(X).$$

Very creatively, MP is called the *even periodic version of complex cobordism.*

Exercise 2.5. Construct the Bott element for MP. (That is, find an element $\beta\in\mathrm{MP}^{-2}(\mathrm{pt})$ such that β is invertible in $\mathrm{MP}^*(\mathrm{pt})$.)

3. Complex oriented spectra

Complex cobordism MU is a remarkable theory. It gives a tremendous amount of structure to any *commutative* ring spectrum that receives a map from it. Because we haven't talked about commutativity of ring spectra (which is a subtle notion), we'll take a different entryway to this story.

If E is a ring spectrum, then we say that its cohomology is *multiplicative*.

Definition 3.1. A multiplicative cohomology theory is *complex oriented* if the map $E^2(\mathbb{CP}^\infty) \to E^2(S^2)$ is surjective. (This map is induced by the inclusion $S^2 \simeq \mathbb{CP}^1 \to \mathbb{CP}^\infty$.)

Choosing a base point of $\mathbb{CP}^\infty$ give a decomposition $E^n(\mathbb{CP}^\infty) \cong \tilde{E}^n(\mathbb{CP}^\infty) \oplus E^n(\mathrm{pt})$, where $\tilde{E}^n$ is reduced cohomology. The same sort of decomposition occurs when we choose a base point for S^2.

Since E is a ring spectrum, $\pi_0 E$ is a ring. The ring structure on $\pi_0 E$ comes with a unit $\bar{t}$, playing the role of 1. Given a map $\theta : E^2(\mathbb{CP}^\infty) \to E^2(S^2)$, we get an induced map $\bar{\theta} : \tilde{E}^2(\mathbb{CP}^\infty) \to \tilde{E}^2(S^2) \cong E^0(\mathrm{pt}) \cong \pi_0 E$. Since the image of $\bar{\theta}$ is a $\pi_0 E$-module, $\bar{\theta}$ is surjective if and only if $\bar{t}$ lies in the image of $\bar{\theta}$. We can thus redefine complex orientation as follows.

Definition 3.2. A *complex orientation* of a multiplicative cohomology theory E is a choice of $t \in \tilde{E}^2(\mathbb{CP}^\infty)$ such that $\bar{\theta}(t) = \bar{t}$ is the canonical generator of $\tilde{E}^2(S^2)$.

Exercise 3.3. Prove that ordinary cohomology is complex orientable by proving that $S^2 \hookrightarrow \mathbb{CP}^\infty$ induces an isomorphism $H^2(\mathbb{CP}^\infty; R) \cong H^2(S^2; R)$ for any commutative ring R.

What is the point of all of this? The space $\mathbb{CP}^\infty$ is the classifying space of complex line bundles, and $E^2(\mathbb{CP}^\infty)$ is the natural home for the first Chern class. Next time, we'll make this rigorous by showing how a complex orientation gives you a theory of first Chern classes for E. We will then study the failure of $c_1(L \otimes L')$ to behave like a group homomorphism. This will give rise to *formal group laws*, which we'll see are also a result of the group structure on an algebraic group.

We'll then see how elliptic integrals give us a bridge between elliptic curves and certain complex oriented, even periodic cohomology theories known as *elliptic cohomology*.

Next time: formal group laws and the $\hat{A}$ and elliptic genera.

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