LECTURE 17: FORMAL GROUP LAWS AND THE $\hat{A}$ AND ELLIPTIC GENERA

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Last time, we said that a ring spectrum E is *complex oriented* if the map $E^2(\mathbb{CP}^\infty) \to E^2(S^2)$ is surjective. The claim was that this gives us enough structure to develop a first Chern class for E. Let's briefly recall the relevant setup.

Choosing a base point of $\mathbb{CP}^\infty$ give a decomposition $E^n(\mathbb{CP}^\infty) \cong \tilde{E}^n(\mathbb{CP}^\infty) \oplus E^n(\mathrm{pt})$, where $\tilde{E}^n$ is reduced cohomology. The same sort of decomposition occurs when we choose a base point for S^2.

Since E is a ring spectrum, $\pi_0 E$ is a ring. The ring structure on $\pi_0 E$ comes with a unit $\bar{t}$, playing the role of 1. Given a map $\theta : E^2(\mathbb{CP}^\infty) \to E^2(S^2)$, we get an induced map $\bar{\theta} : \tilde{E}^2(\mathbb{CP}^\infty) \to \tilde{E}^2(S^2) \cong E^0(\mathrm{pt}) \cong \pi_0 E$. Since the image of $\bar{\theta}$ is a $\pi_0 E$-module, $\bar{\theta}$ is surjective if and only if $\bar{t}$ lies in the image of $\bar{\theta}$. We can thus redefine complex orientation as follows.

Definition 0.1. A *complex orientation* of a multiplicative cohomology theory E is a choice of $t \in \tilde{E}^2(\mathbb{CP}^\infty)$ such that $\bar{\theta}(t) = \bar{t}$ is the canonical generator of $\tilde{E}^2(S^2)$.

Exercise 0.2. Prove that ordinary cohomology is complex orientable by proving that $S^2 \hookrightarrow \mathbb{CP}^\infty$ induces an isomorphism $H^2(\mathbb{CP}^\infty; R) \cong H^2(S^2; R)$ for any commutative ring R.

What is the point of all of this? The space $\mathbb{CP}^\infty$ is the classifying space of complex line bundles, and $E^2(\mathbb{CP}^\infty)$ is the natural home for the first Chern class. Today, we'll make this rigorous by showing how a complex orientation gives you a theory of first Chern classes for E. We will then study the failure of $c_1(L \otimes L')$ to behave like a group homomorphism. This will give rise to *formal group laws*, which we'll see are also a result of the group structure on an algebraic group.

We'll then see how elliptic integrals give us a bridge between elliptic curves and certain complex oriented, even periodic cohomology theories known as *elliptic cohomology.*

The key result of having a complex orientation t is an isomorphism of the form

$$E^*(\mathbb{CP}^\infty) \cong E^*(\mathrm{pt})[\![t]\!].$$

(Compare to $H^*(\mathbb{CP}^\infty; \mathbb{Z}) \cong \lim \mathbb{Z}[x]/(x^n)$ with $|x| = 2$.) Now $\mathbb{CP}^\infty$ is the classifying space of line bundles, and its cohomology is generated by the first Chern class. So $c_1 := t$, which is how a complex orientation gives us c_1.

As a cohomology class, c_1 gives us a map from bundles over $\mathbb{CP}^\infty$ (i.e. line bundles over a space X) to $E^*(\mathrm{pt})$. If we want to compute $c_1(L \otimes L')$, we use the universal tensor product

$$\begin{array}{ccc} \xi \otimes \xi & \longrightarrow & \xi \\ \downarrow & & \downarrow \\ \mathbb{CP}^\infty \times \mathbb{CP}^\infty & \longrightarrow & \mathbb{CP}^\infty. \end{array}$$

On cohomology, this induces a map

$$E^*(\mathbb{CP}^\infty) \to E^*(\mathbb{CP}^\infty) \otimes E^*(\mathbb{CP}^\infty)$$
$$c_1 \mapsto F(x_1, x_2).$$

Here, $F(x_1, x_2) \in E^*(\mathrm{pt})[\![x_1, x_2]\!]$ is a formal power series satisfying $c_1(L \otimes L') = F(c_1(L), c_1(L'))$.

Exercise 0.3. Using properties of c_1, prove that:

(i) $F(x, 0) = F(0, x) = x$.

(ii) $F(x, y) = F(y, x)$.

(iii) $F(F(x, y), z) = F(x, F(y, z))$.

1. Formal group laws

While formal group laws arise for us in expanding $c_1(L \otimes L')$, they derive their name from algebraic groups. In particular, a *formal group law* is the power series expansion of the group operation of an algebraic group in a neighborhood of the identity.

Example 1.1. The identity of $\mathbb{G}_a$ is 0. In a neighborhood of 0, the group law is given by $(0+x)+(0+y) = x+y$. The corresponding *additive formal group law* is $f(x, y) = x+y$.

The identity of $\mathbb{G}_m$ is 1. In a neighborhood of 1, the group law is given by $(1+x)\cdot(1+y) = 1+x+y+xy$. The corresponding *multiplicative formal group law* is $f(x, y) = x+y+xy$.

In an effort to glean from algebraic groups the key properties of a formal group law, we are led to the following definition. (This is just the definition for 1-dimensional commutative formal group laws. One can also define n-dimensional formal group laws, but we won't need these.)

Definition 1.2. A *formal group law* over a ring R is a formal power series $F(x, y) \in R[\![x, y]\!]$ satisfying:

(i) $F(x, 0) = F(0, x) = x$.

(ii) $F(x, y) = F(y, x)$.

(iii) $F(F(x, y), z) = F(x, F(y, z))$.

These are the identity, commutativity, and associativity axioms under the guise of the power series expansion.

Example 1.3. In ordinary cohomology, $c_1(L \otimes L') = c_1(L) + c_1(L')$. So if E is the spectrum representing oriented cohomology, then its associated formal group law is the additive formal group law.

Example 1.4. In K-theory, we have $c_1(L) = [L] - 1 \in K^0(X)$ for any line bundle $L \to X$. Thus

$$\begin{aligned} c_1(L \otimes L') &= [L \otimes L'] - 1 \\ &= ([L]-1)([L']-1) + [L] + [L'] - 2 \\ &= ([L]-1)([L']-1) + ([L]-1) + ([L']-1) \\ &= c_1(L) + c_1(L') + c_1(L)c_1(L'). \end{aligned}$$

So the formal group law associated to K-theory is the multiplicative group law.

2. Genera

We just learned that the additive and multiplicative formal group laws come from $\mathbb{G}_a$ and $\mathbb{G}_m$, respectively. But there are more interesting 1-dimensional commutative algebraic groups out there, like elliptic curves. What do their associated formal group laws look like?

Well, the easiest formulation of the group law on an elliptic curve comes from writing it as a quotient $\mathbb{C}/\Lambda$. The group structure on $\mathbb{C}/\Lambda$ comes from the group structure on Λ, and we recover Λ from its elliptic curve by taking period integrals over homology generators. Recall that we could write these period integrals purely in terms of one parameter, giving us integrals of the form

$$\int \frac{\mathrm{d}x}{\sqrt{4x^3 - ax - b}}.$$

We're going to change this notation a bit to match up with conventions in topology. A *Jacobi elliptic curve* is an elliptic curve of the form

$$J := \mathbb{V}(y^2 = \varepsilon x^4 - 2\delta x^2 + 1),$$

where $\varepsilon(\delta^2 - \varepsilon)^2 \neq 0$.

Exercise 2.1. Try writing a Jacobi elliptic curve in Weierstrass form to verify that these are indeed elliptic curves.

The lattice associated to a Jacobi elliptic curve is given by period integrals of the form

$$\int \frac{\mathrm{d}x}{\sqrt{\varepsilon x^4 - 2\delta x^2 + 1}}.$$

Under this parameterization, the identity of the elliptic curve is given at $x = 0$. Moreover, we add two points on our curve by adding in $\mathbb{C}$ modulo the lattice Λ, and then returning back to our elliptic curve. So in the neighborhood of the identity, points on our elliptic are added according to

$$\int_0^z \frac{\mathrm{d}x}{\sqrt{\varepsilon x^4 - 2\delta x^2 + 1}} + \int_0^{z'} \frac{\mathrm{d}x}{\sqrt{\varepsilon x^4 - 2\delta x^2 + 1}}.$$

In other words, the formal group law $F_J(z, z')$ associated to the Jacobi elliptic curve J satisfies the equation

$$\int_0^z \frac{\mathrm{d}x}{\sqrt{\varepsilon x^4 - 2\delta x^2 + 1}} + \int_0^{z'} \frac{\mathrm{d}x}{\sqrt{\varepsilon x^4 - 2\delta x^2 + 1}} = \int_0^{F_J(z,z')} \frac{\mathrm{d}x}{\sqrt{\varepsilon x^4 - 2\delta x^2 + 1}}.$$

Remark 2.2. At this point, you should think, "This is all fine and good, but we don't even have a closed form for these elliptic integrals. How are we ever supposed to get a closed form for F_J?" Behold the terrible might of Euler.

Theorem 2.3 (Euler)**.**

$$F_J(x_1, x_2) = \frac{x_1\sqrt{\varepsilon x_1^4 - 2\delta x_1^2 + 1} + x_2\sqrt{\varepsilon x_2^4 - 2\delta x_2^2 + 1}}{1 - \varepsilon x_1^2 x_2^2}.$$

Exercise 2.4. Expand out $F_J(x_1, x_2)$ as a power series and verify that it satisfies the axioms of a formal group law.

Remark 2.5. While our narrative of using period integrals required that our elliptic curves be defined over $\mathbb{C}$, Euler's formula for the formal group law holds for any elliptic curve defined over $\mathbb{Z}[\delta, \varepsilon]$. Moreover, the previous exercise shows you that you can write out the formal group law as a power series over $\mathbb{Z}[\frac{1}{2}, \delta, \varepsilon]$.

Remark 2.6. Recall (from the first few weeks of class) that

$$\log_J(z) = \int_0^z \frac{\mathrm{d}x}{\sqrt{\varepsilon x^4 - 2\delta x^2 + 1}}$$

is the logarithm of the *elliptic genus*. The corresponding genus satisfies $\Phi_J(\mathbb{CP}^2) = \delta$ and $\Phi_J(\mathbb{HP}^2) = \varepsilon$. By keeping δ and ε as variables (satisfying $\varepsilon(\delta^2 - \varepsilon)^2 \neq 0$) gives us the *universal elliptic genus*.

However, if we specialize ε and δ such that $\varepsilon(\delta^2 - \varepsilon)^2 = 0$, we get a singular elliptic curve. When $\delta = \varepsilon = 1$, $\log_J$ simplifies to

$$\log_L(z) = \int_0^z \frac{\mathrm{d}x}{1 - x^2},$$

which you may recall as the logarithm of the L-genus (i.e. signature genus). When $\delta = -1/8$ and $\varepsilon = 0$, we get

$$\log_A(z) = \int_0^z \frac{\mathrm{d}x}{\sqrt{1 + (x/2)^2}}.$$

This is the logarithm of the $\hat{\mathcal{A}}$-genus, which is the index of the Dirac operator on spin manifolds. This is an interesting and important part of the story that we unfortunately don't have enough time to cover. (But writing about spin manifolds, the Dirac operator, and the $\hat{\mathcal{A}}$-genus would make a good written project!)

Exercise 2.7. What singularity types do the elliptic curves related to the L-genus and $\hat{\mathcal{A}}$-genus have?

3. Elliptic cohomology

We saw how every complex oriented cohomology theory yields a formal group law. So what is the cohomology theory underlying the Jacobi formal group law? The *Landweber exact functor theorem* gives a list of sufficient criteria for a formal group law to determine its cohomology theory in a systematic way. Landweber, Ravenel, and Stong showed that elliptic formal group laws satisfy these criteria, which allows one to write down a formula for *elliptic cohomology*:

$$\mathrm{Ell}^*(X) = \mathrm{MP}^*(X) \otimes_{\mathrm{MP}^*(\mathrm{pt})} \mathbb{Z}[\tfrac{1}{2}, \delta, \varepsilon, \Delta^{-1}],$$

where $\Delta = \varepsilon(\delta^2 - \varepsilon)^2$ and MP is the periodization of MU. Here's an abstracted definition of elliptic cohomology:

Definition 3.1. An *elliptic cohomology theory* consists of:

(i) An even periodic multiplicative cohomology theory.

(ii) An elliptic curve C over a ring R.

(iii) An isomorphism $E^0(\mathrm{pt}) \cong R$ and an isomorphism[1] of the formal group law associated to E with the formal group law associated to C.

By keeping δ and ε as parameters, we get a "universal" elliptic formal group law and elliptic genus. It is natural to wonder if there is a spectrum whose cohomology is this "universal" elliptic cohomology theory. In order to make this precise, one needs to work with the universal elliptic curve, namely the elliptic curve from which all elliptic curves are pulled back.

We've already discussed how automorphisms prevent such an object from occuring in any naïve construction of the moduli space of elliptic curves. But one key feature of working with stacks is that we in fact get a universal elliptic curve. So if we want to build a spectrum that fills the role of the "universal elliptic spectrum," we need a way to mimic the construction of $\mathcal{M}_{1,1}$ in the world of spectra. This is why we'll need to talk about derived algebraic geometry. The eventual payoff will be tmf, the universal elliptic spectrum.

[1]I haven't told you what an morphism of formal group laws is, let alone an isomorphism. It's a good exercise to work out these definitions for yourself.

Next time: Witten genus.

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