LECTURE 18: ELLIPTIC COHOMOLOGY

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1. Elliptic cohomology

Last time, we saw how every complex oriented cohomology theory yields a formal group law (the class $t \in \tilde{E}^2(\mathbb{CP}^\infty)$ plays the role of c_1, and the formal group law is given by expanding $c_1(L \otimes L')$ as a power series in $c_1(L)$ and $c_1(L')$). So what is the cohomology theory underlying the Jacobi formal group law? The *Landweber exact functor theorem* gives a list of sufficient criteria for a formal group law to determine its cohomology theory in a systematic way.

To talk about Landweber exactness, we first have to meet the Lazard ring. We're going to tell this story in the periodic setting, but there's also a non-periodic version (in which you should replace MP below with MU).

Definition 1.1. The *Lazard ring* $\mathbb{L}$ is the ring $\mathbb{Z}[\{a_{ij}\}_{i,j\in\mathbb{N}}]$, subject to the relations:

(i) $a_{10} = a_{01} = 1$.

(ii) $a_{ij} = a_{ji}$.

(iii) $\ell(x, \ell(y, z)) = \ell(\ell(x, y), z)$, where $\ell(x, y) := \sum_{i,j} a_{ij}x^iy^j \in \mathbb{L}[\![x, y]\!]$.

As you can see, $\mathbb{L}$ looks like it is cooked up to be the universal ring with a formal group law. This is indeed the case.

Theorem 1.2. *If R is a ring with a formal group law $f(x, y) \in R[\![x, y]\!]$, then there is a unique ring homomorphism $\phi : \mathbb{L} \to R$ such that $\phi_*\ell = f$.*

Exercise 1.3. Prove Theorem 1.2.

We've actually seen $\mathbb{L}$ before: $\mathbb{L} \cong \mathrm{MP}^*(\mathrm{pt})$, where MP is the periodization of MU. So given a formal group law f over R, we'd like to build a cohomology theory. It would be really nice if we could just "change scalars" from the cohomology theory coming from the universal formal group law:

$$E_f^*(X) := \mathrm{MP}^*(X) \otimes_{\mathrm{MP}^*(\mathrm{pt})} R.$$

In order for this to give us a spectrum, the assignment $X \mapsto E_f^*(X)$ needs to satisfy the properties of a generalized cohomology theory. A sufficient criterion is for $X \mapsto E_f^*(X)$ to be exact, i.e. for R to be a flat $\mathrm{MP}^*(\mathrm{pt})$-module. But this is a very restrictive condition. A weaker sufficient condition is *Landweber exactness*:

Definition 1.4. Let $f(x, y)$ be a formal group law over R. Using the associativity axiom, we can take $[p] \cdot x := f(f(\cdots(f(x, x), x)))$ to denote adding x to itself p times. When p is a prime, let $v_{i,p}$ denote the coefficient of x^{p^i} in $[p] \cdot x$.

The formal group law f is called *Landweber exact* if $v_{0,p}, \ldots, v_{i,p}$ is a regular sequence in R for all p and all i.

Landweber, Ravenel, and Stong showed that elliptic formal group laws satisfy the criteria of the Landweber exact functor theorem, which allows one to write down a formula for *elliptic cohomology*:

$$\mathrm{Ell}^*(X) = \mathrm{MP}^*(X) \otimes_{\mathrm{MP}^*(\mathrm{pt})} \mathbb{Z}[\tfrac{1}{2}, \delta, \varepsilon, \Delta^{-1}],$$

where $\Delta = \varepsilon(\delta^2 - \varepsilon)^2$ and MP is the periodization of MU. Here's an abstracted definition of elliptic cohomology:

Definition 1.5. An *elliptic cohomology theory* consists of:

(i) An even periodic multiplicative cohomology theory.

(ii) An elliptic curve C over a ring R.

(iii) An isomorphism $E^0(\mathrm{pt}) \cong R$ and an isomorphism[1] of the formal group law associated to E with the formal group law associated to C.

A spectrum that determines an elliptic cohomology theory is called an *elliptic spectrum.*

By keeping δ and ε as parameters, we get a "universal" elliptic formal group law and elliptic genus. It is natural to wonder if there is a spectrum whose cohomology is this "universal" elliptic cohomology theory. In order to make this precise, one needs to work with the universal elliptic curve, namely the elliptic curve from which all elliptic curves are pulled back.

We've already discussed how automorphisms prevent such an object from occuring in any naïve construction of the moduli space of elliptic curves. But one key feature of working with stacks is that we in fact get a universal elliptic curve. So if we want to build a spectrum that fills the role of the "universal elliptic spectrum," we need a way to mimic the construction of $\mathcal{M}_{1,1}$ in the world of spectra. This is why we'll need to talk about derived algebraic geometry. The eventual payoff will be tmf, the universal elliptic spectrum.

2. Lifting genera

We've mentioned the $\hat{\mathcal{A}}$-genus a few times in this course, always defining it in terms of its logarithm. The only other way to define the $\hat{\mathcal{A}}$ genus is as the index of the Dirac

[1] I haven't told you what an morphism of formal group laws is, let alone an isomorphism. It's a good exercise to work out these definitions for yourself.

operator. The Dirac operator $\not{D}$ is a square root of (minus[2]) the Laplace operator

$$\Delta = -\sum_i \frac{\partial^2}{\partial x_i^2},$$

which we saw in an exercise earlier in the class (about isospectral but non-isometric tori).

Exercise 2.1. The Dirac operator takes the form

$$\not{D} = \sum_{i=1}^{n} \gamma_i \frac{\partial}{\partial x_i}$$

and satisfies $\not{D}^2 = -\Delta$.

(i) Calculate $\not{D}$ when $n = 1$.

(ii) Show that for $n > 1$, the coefficients γ_i cannot be complex numbers.

(iii) For $n > 1$, find matrices γ_i that give you a solution to $\not{D}^2 = -\Delta$. Are your solutions unique?

Definition 2.2. Once you've constructed the correct Dirac operator, you can consider its action on the tangent space of a manifold M. The *index* of $\not{D}$ is the value $\operatorname{ind}(\not{D}) = \dim\ker\not{D} - \dim\operatorname{coker}\not{D}$. Define the $\hat{\mathcal{A}}$-genus of M as $\operatorname{ind}_M(\not{D})$.

Remark 2.3. The previous definition is a little bit of a gentle lie, since the $\hat{\mathcal{A}}$-genus need not be an integer on arbitrary manifolds. The Atiyah–Singer index theorem implies that $\hat{\mathcal{A}}$ is an index on spin manifolds, and that

$$\hat{\mathcal{A}} : \Omega_*^{\mathrm{Spin}} \to \mathbb{Z}$$

is a genus.

Next time: The Witten genus.

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[2]The sign convention is a physical artifact and isn't too important.