LECTURE 19: THE WITTEN GENUS

STEPHEN MCKEAN

1. Lifting the $\hat{\mathcal{A}}$-genus

The $\hat{\mathcal{A}}$-genus is defined as the index of a certain Dirac operator.[1] We won't properly define the relevant operator, but I will just say the key words for those who are interested. Given a manifold M with spin bundle $S \to M$, we define $\not{D}$ on S locally as

$$\not{D}s(x) = \sum_{i=1}^{n} e_i(x)\tilde{\Gamma}_{e_i(x)}s(x),$$

where $x \in M$, the tangent vectors $e_1(x), \ldots, e_n(x)$ form an orthonormal basis of T_xM, the *spin connection* $\tilde{\Gamma}$ is the lift to S of the Levi-Cevita connection on TM, and $s : M \to S$ is a section.

This lets us superficially describe the construction of the Atiyah–Bott–Shapiro orientation at the level of

$$\mathrm{MSpin}_* \to \mathrm{KO}_*.$$

Points in MSpin_n are n-manifolds equipped with spin structure. Atiyah–Bott–Shapiro rephrase this spin structure as a bundle of $\mathrm{Cliff}(\mathbb{R}^n)$-$\mathrm{Cliff}(TM)$ bimodules, known as the *spinor bundle* $\mathrm{Spinor}(M)$ of M.

There are many ways to construct real K-theory, but the relevant one for us again goes through Clifford algebras. A point in KO_n is a real Hilbert space equipped with a $\mathrm{Cliff}(\mathbb{R}^n)$-action and an odd skew-adjoint[2] $\mathrm{Cliff}(\mathbb{R}^n)$-linear Fredholm[3] operator.

Given a metric on M, we can define the Levi-Cevita connection on TM and hence the spin connection $\tilde{\Gamma}$. The Atiyah–Bott–Shapiro orientation sends M to the Hilbert space of L^2-sections of $\mathrm{Spinor}(M)$ with Fredholm operator $\not{D}$. One needs to check that $\not{D}$ is odd, skew-adjoint, Clifford-linear, and so on, but this gives you an idea for constructing the necessary maps

$$\mathrm{MSpin}_n \to \mathrm{KO}_n$$

of topological spaces. It requires a little extra work to stitch this all together and get a map of spectra

$$\mathrm{MSpin} \to \mathrm{KO},$$

but this can be done. This gives an indication of how the ABS orientation lifts the $\hat{\mathcal{A}}$-genus to the level of spectra: $\mathrm{MSpin} \to \mathrm{KO}$ is given by $M \mapsto \not{D}$, so we just need to

[1] Well, almost. Dirac operators are square roots of the Laplacian, while the $\hat{\mathcal{A}}$-genus is the index of a square root of the Laplacian shifted by a curvature operator.

[2] $A^* = -A$.

[3] A bounded linear operator between Banach spaces with finite dimensional kernel and cokernel.

realized "taking the index" as $\pi_0(M \mapsto \not{D})$. This follows from the Atiyah–Singer index theorem, which states that the index is equal to

$$(-1)^n \int_M \mathrm{ch}(\not{D})\mathrm{Td}(M),$$

where ch is the Chern character[4] and Td is the Todd class.

2. The Witten genus

The point of the first section was to convince you that lifting genera to the level of spectra is a tricky business. There is no canonical procedure for giving these lifts. Instead, we need some sort of thorough understanding of the analysis underlying the relevant geometry (e.g. the Atiyah–Singer index theorem, as well as the role of Clifford algebras in spin geometry and K-theory).

The rest of today's lecture is devoted to trying to explain what the Witten genus is and where it comes from. To really do this justice, we would need to talk about partition functions in quantum field theory, which we can't possibly cover in this class. But hopefully we will get some intuition, and maybe even see why modular forms should show up.

Very roughly, the partition function of an n-dimensional quantum field theory is a function on the space of maps $T^n \to M$, where T^n is an n-torus and M is some manifold. Witten showed that for a certain 1-dimensional qft, the partition function of $S^1 \to M$ is $\mathrm{ind}(\not{D}_M)$ whenever M is spin [Wit82]. That is, one can reformulate the $\hat{\mathcal{A}}$-genus as a partition function.

Witten later explained what should happen for a certain 2-dimensional qft [Wit87]. The partition function maps from the space of maps $T^2 \to M$, so we want to study the infinite-dimensional manifold of smooth maps $T^2 \to M$. This is closely related to the *free loop space* of M.

Definition 2.1. Given a manifold M, the *free loop space* $LM := C^\infty(S^1, M)$ is the space of smooth maps from $S^1 \to M$. Just as with the *pointed* loop space $\Omega_x M$, we need to do some functional analysis to put a topology on this set. It's a bit more work to show that LM can be viewed as an infinite-dimensional manifold.

Exercise 2.2. Prove that the manifold of smooth maps $T^2 \to M$ is diffeomorphic to the manifold of smooth maps $S^1 \to LM$.

Remark 2.3. Passing to the loop space is a dimension reduction technique in quantum field theory. Indeed, a 2-dimensional qft on M determines a 1-dimensional qft on LM by sending $\{I \times S^1 \to M\} \xrightarrow{\cong} \{I \to LM\}$. By Witten's work in dimension 1, the partition function of such a 2-dimensional qft on M should be the partition function of a 1-dimensional qft on LM, namely the index of a Dirac operator on LM. But there is no known theory of Dirac operators on LM, because LM is infinite dimensional.

[4] Modulo several technical intermediating steps.

We're going to press forward and pretend that we can define $\not{D}_{LM}$. For this narrative, it will be relevant to talk about spin structures on loop spaces.

Recall that $\mathrm{SO}(n) \to \mathrm{O}(n)$ is a connected cover, i.e. a covering map that trivializes $\pi_0(\mathrm{O}(n))$. Similarly $\mathrm{Spin}(n) \to \mathrm{SO}(n)$ is a universal cover, or a covering map that trivializes $\pi_1(\mathrm{SO}(n))$. Continuing this trend, we get the *Whitehead tower*

$$\cdots \to \mathrm{String}(n) \to \mathrm{Spin}(n) \to \mathrm{SO}(n) \to \mathrm{O}(n),$$

where each map is a covering that kills the lowest non-trivial homotopy group. Given any manifold M of dimension n, we have an action of $\mathrm{GL}_n(\mathbb{R})$ on TM. We get G-structure on M (for some Lie group G) if the *structure group* $\mathrm{GL}_n(\mathbb{R})$ admits a reduction $\mathrm{GL}_n(\mathbb{R}) \times G$ that acts on TM. A manifold is *spin* if we can take $G = \mathrm{Spin}(n)$. There is a subtlety to repeating this story for LM, but here are two ways to define LM being spin.

Definition 2.4. Let M be a manifold of dimension n. Then LM is *spin* if either of the following equivalent conditions hold:

(i) The class $\frac{1}{2}p_1(M) \in H^4(M;\mathbb{Z})$ vanishes, where $\frac{1}{2}p_1$ is a square root of the first Pontryagin class.

(ii) The structure group of M admits a lift to $\mathrm{String}(n)$ (that is, M is *string*).

Now back to this hypothetical Dirac operator. We can enhance an operator $\not{D}_M$ by a vector bundle $V \to M$ and compute the index

$$\mathrm{ind}(\not{D}_M \otimes V) = \dim\ker(\not{D}_M \otimes V) - \dim\mathrm{coker}(\not{D}_M \otimes V).$$

The Atiyah–Singer index theorem works in this generality, computing

$$\mathrm{ind}(\not{D}_M \otimes V) = \int_M \mathrm{ch}(V)[\hat{\mathcal{A}}(M)],$$

where $[\hat{\mathcal{A}}(M)] \in H^*_{\mathrm{dR}}(M;\mathbb{R})$ is called the $\hat{\mathcal{A}}$-class.[5]

Atiyah–Singer can be strengthened even further to an equivariant version — this is good, because LM admits a natural smooth S^1-action. Because this action is smooth, we get induced S^1-actions on $\mathrm{Spinor}(LM)$ and on $V \to LM$. Moreover, $\ker(\not{D}_{LM} \otimes V)$ and $\mathrm{coker}(\not{D}_{LM} \otimes V)$ are invariant under the S^1-action, so these are S^1-representations (instead of just vector spaces). Given an S^1-representation R, let R_m denote the subspace on which $s \in S^1$ acts as multiplication by s^m. Then the *equivariant index* of $\not{D}_{LM} \otimes V$ is

$$\mathrm{ind}^{S^1}(\not{D}_{LM} \otimes V) = \sum_{m\in\mathbb{Z}} \left(\dim\ker(\not{D}_{LM} \otimes V)_m - \dim\mathrm{coker}(\not{D}_{LM} \otimes V)_m\right) q^m \in \mathbb{Z}[q^{\pm 1}].$$

Using physics as a guide, Witten predicted a formula for this hypothetical $\mathrm{ind}^{S^1}(\not{D}_{LM})$ [Wit88]: $\mathrm{ind}^{S^1}(\not{D}_{LM}) = \eta(q)^{-n} \cdot \Phi_{\mathrm{Wit}}(M)$, where $\eta(q) = q^{1/24}\prod_{n\geq 1}(1-q^n)$ and

$$\Phi_{\mathrm{Wit}}(M) := \int_M [\hat{\mathcal{A}}(M)] \wedge \mathrm{ch}\left(\bigotimes_{m\geq 1} \mathrm{Sym}_{q^m}((TM - \mathbb{R}^n)\otimes\mathbb{C})\right).$$

[5]The $\hat{\mathcal{A}}$-class computes the $\hat{\mathcal{A}}$-genus: $\int_M [\hat{\mathcal{A}}(M)] = \hat{\mathcal{A}}(M)$.

Here, $\mathrm{Sym}_t(V) = \mathbb{C} \oplus V \cdot t \oplus \mathrm{Sym}^2(V) \cdot t^2 \oplus \cdots$.

Definition 2.5. The power series $\Phi_{\mathrm{Wit}}(M) \in \mathbb{R}[\![q]\!]$ is the *Witten genus* of a closed orientable manifold M.

Proposition 2.6. *If M is spin, then $\Phi_{\mathrm{Wit}}(M) \in \mathbb{Z}[\![q]\!]$.*

Proof. When M is spin, the Atiyah–Singer index theorem tells us that

$$\begin{aligned}\Phi_{\mathrm{Wit}}(M) &= \int_M [\hat{\mathcal{A}}(M)] \wedge \mathrm{ch}\left(\bigotimes_{m\geq 1} \mathrm{Sym}_{q^m}((TM - \mathbb{R}^n) \otimes \mathbb{C})\right) \\ &= \sum_{i\geq 0} \mathrm{ind}(\not{D}_M \otimes V_i) \cdot q^i,\end{aligned}$$

where V_i is the vector space coefficient of q^i in

$$\bigotimes_{m\geq 1} \mathrm{Sym}_{q^m}((TM-\mathbb{R}^n)\otimes\mathbb{C}) = \mathbb{C}\oplus((TM-\mathbb{R}^n)\otimes\mathbb{C})\cdot q\oplus(\mathrm{Sym}^2((TM-\mathbb{R}^n)\otimes\mathbb{C})\oplus\mathbb{C})\cdot q^2\oplus\cdots.$$

Since each of these indices is an integer, $\Phi_{\mathrm{Wit}}(M) \in \mathbb{Z}[\![q]\!]$. □

Witten and Zagier each showed that if M is string (i.e. if LM is spin),[6] then $\Phi_{\mathrm{Wit}}(M) \in \mathbb{Z}[\![q]\!]$ is a modular form of weight $n/2$ [Wit88; Zag88]. So if we wanted to write down the Witten genus as a graded ring homomorphism (and if we wanted to ignore torsion), we would have

$$\Phi_{\mathrm{Wit}} : \Omega_*^{\mathrm{String}} \otimes \mathbb{Q} \to \mathbb{Q}[G_4, G_6].$$

Exercise 2.7. Construct an 8-dimensional string manifold M and a 12-dimensional string manifold N such that $\Phi_{\mathrm{Wit}}(M) = a \cdot G_4$ and $\Phi_{\mathrm{Wit}}(N) = b \cdot G_6$.[7]

The upshot of this previous exercise is that Φ_{Wit} is a surjective homomorphism $\Omega_*^{\mathrm{String}} \otimes \mathbb{Q} \to \mathbb{Q}[G_4, G_6]$.

If we don't want to ignore torsion, we still get a graded ring homomorphism

$$\Phi_{\mathrm{Wit}} : \Omega_*^{\mathrm{String}} \to \mathrm{MF}_*.$$

Since $\mathrm{String}(n)$ form a family of Lie groups, we can construct the Thom spectrum MString.

Question 2.8. Is there some ring spectrum X such that Φ_{Wit} lifts to a ring spectrum map

$$\mathrm{MString} \to X?$$

Topological modular forms will be the answer to this question.

Next time: E_∞-ring spectra and derived algebraic geometry.

[6]That is, we need $\frac{1}{2}p_1(M) = 0$. In fact, Witten and Zagier show that if $p_1(M) = 0$, then the Witten genus is a modular form.

[7]I don't know how to do this, but Stong claims it is an "easy" application of surgery.

References

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Department of Mathematics, Harvard University

Email address: smckean@math.harvard.edu

URL: shmckean.github.io