LECTURE 2: KEPLER'S LAWS AND ELLIPTIC FUNCTIONS

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Last time, we learned a little bit about genera (i.e. cobordism invariants). As we discussed, the discovery of topological modular forms was catalyzed by the introduction of the elliptic genus and its connection to elliptic cohomology. Because the term *elliptic* seems to show up everywhere, today's lecture will be all about *elliptic functions*. I want to begin by explaining why elliptic functions are an inevitable piece of scientific history.[1]

1. Kepler's laws of planetary motion

The stars have long inspired the human mind. While most lights in the heavens follow a clear periodic motion, the planets wander. For millennia, the motion of the planets was a great mystery. A revolution in physics came about with Copernicus's theory of heliocentrism. Copernicus posited that the planets follow circular orbits around the sun, with the sun lying at the center.

Heliocentrism was revised by Kepler, who used Tycho Brahe's extensive astronomical measurements to show that while Mars does indeed orbit the sun, its orbit is not circular and the sun does not lie at the center. Instead, Mars's orbit is an ellipse, and the sun lies at one of the foci. These observations led Kepler to propose three laws of planetary motion:

Law 1.1 (Kepler)**.**

(1) The orbit of a planet around the sun is an ellipse, with the sun at one of the foci.

(2) A line segment joining the sun and planet sweeps out equal areas during equal intervals of time.

(3) The square of a planet's orbital period is proportional to the cube of its orbit's semi-major axis.

Question 1.2. It turns out that all of the inner planets' orbits are almost circular, but Mars's orbit is the most eccentric these. Would it have been possible to recognize a less eccentric orbit as non-circular with the tools of Kepler's time? Would more precise tools for astronomical measurements have been developed without a knowledge of Kepler's laws? How might this have changed the history of science had we appeared on Mars instead of Earth?

[1] When learning something new, I often ask myself: was this discovered by coincidence? Or would a different society on a different world eventually come up with the same concept?

Before we let this turn into an existential crisis, let's see how Newton's laws of motion imply Kepler's laws of planetary motion.

(1) Suppose we have two bodies of masses M and m. Fix our frame of reference to the larger body, so that the smaller body is moving with velocity v at distance r. Newton then gives us the total energy of the system as
$$E = \frac{mv^2}{2} - \frac{GMm}{r}.$$
Now we rewrite v in polar coordinates. The radial and tangential components of v are orthogonal, so $v^2 = r'^2 + r^2\theta'^2$.

The angular momentum of the smaller body is $L = r \times mv$. Since the vectors r and r' are parallel, we have $r \times r' = 0$. On the other hand, the vectors r and $r\theta'$ are orthogonal, so we find that $L = mr^2\theta'$. Substituting $\theta' = \frac{L}{mr^2}$, we get
$$\begin{aligned} E &= \frac{m}{2}\left(r'^2 + r^2\left(\frac{L}{mr^2}\right)^2\right) - \frac{GMm}{r} \\ &= \frac{mr'^2}{2} + \frac{L^2}{2mr^2} - \frac{GMm}{r}. \end{aligned}$$
Solve for r' and use the non-obvious substitutions $a = \frac{L^2}{GMm^2}$ and $e^2 = 1 + \frac{2Ea}{GMm}$ to find
$$r' = \frac{L}{m}\left(\frac{e^2}{a^2} - \left(\frac{1}{r} - \frac{1}{a}\right)^2\right)^{1/2}.$$
Now let $\rho = 1/r$, and note that $r' = -\frac{1}{\rho^2}\rho'$. Since
$$\begin{aligned} \theta &= \int \mathrm{d}\theta \\ &= \int \frac{L}{mr^2}\,\mathrm{d}t \\ &= \int \frac{L\rho^2}{m}\frac{\mathrm{d}t}{\mathrm{d}\rho}\,\mathrm{d}\rho \\ &= -\int \frac{L}{mr'}\,\mathrm{d}\rho, \end{aligned}$$
we find that
$$\begin{aligned} \theta &= -\int \frac{1}{\sqrt{e^2/a^2 - (\rho - 1/a)^2}}\,\mathrm{d}\rho \\ &= \arccos\left(\frac{\rho - 1/a}{e/a}\right). \end{aligned}$$
In other words, $r = \frac{a}{1+e\cos\theta}$, which is an ellipse with one focus at the origin.

(2) The infinitesimal area $\mathrm{d}A$ swept out by the planet is a right triangle with legs r and $\mathrm{d}r$, so we have $\mathrm{d}A = \frac{r}{2} \times \mathrm{d}r$. It follows that rate of area swept out is $A' = \frac{r}{2} \times r'$. To show that this rate of area sweeping is constant, we need to show

that $A'' = 0$. By the product rule, we have $A'' = \frac{1}{2}(r' \times r' + r \times r'')$. The first term vanishes, but the second term need not vanish in general. It happens to vanish for planetary motion, because Newton's laws imply that the acceleration of the planet due to gravity is parallel to the line through the sun and planet. Thus $r \times r'' = 0$, so $A'' = 0$.

Exercise 1.3. Derive Kepler's third law from Newton's laws of motion. (Hint: the area of an ellipse with semi-major axis a and eccentricity e is $A = \pi a\sqrt{1-e^2}$. At the end of the day, you will find that $T^2 = \frac{4\pi^2 a^3}{GM}$, where T is the orbital period.)

With Kepler's laws in hand, we can now predict the motion of the planets in the sky. This is one of the greatest scientific achievements in history — humanity long thought[2] that the motion of the planets determined our fate. Kepler's laws predict the future motion of the planets and thus any fate tied to their motion.

But what do we want to do with this information? The most basic questions we can ask about motion are how far, and how fast? For example, how much distance will the earth cover during this semester? Thanks to calculus, we know how to formulate the distance travelled along an arc. For an ellipse $(a\cos t, b\sin t)$, this takes the form

$$\int_{t_0}^{t_1} \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}\ \mathrm{d}t. \tag{1.1}$$

If you're averse to trigonometry, you could use the substitution $x = \sin t$ and write

$$\begin{aligned}\int_{t_0}^{t_1} \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}\ \mathrm{d}t &= \int_{t_0}^{t_1} \sqrt{b^2 + (a^2 - b^2)\sin^2 t}\ \mathrm{d}t \\ &= \int_{\sin t_0}^{\sin t_1} \sqrt{b^2 + (a^2-b^2)x^2}\ \mathrm{d}(\sin^{-1} x) \\ &= b\int_{\sin t_0}^{\sin t_1} \frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}\ \mathrm{d}x,\end{aligned}$$

where $k = 1 - (\frac{a}{b})^2$.

Question 1.4. Can you solve this integral?

Exercise 1.5. Write a program to numerically integrate Equation 1.1 up to specified precision.

2. Elliptic integrals

Equation 1.1 is an example of an *elliptic integral*. The reason for the name is abundantly clear. There are many other types of elliptic integrals, not all of which have anything to do with ellipses. Historically, elliptic integrals arose as solutions to interesting physical problems. For example, while trigonometric functions can be used to model pendulum motion when the displacement angle is small, but larger displacements require elliptic

[2]And still does, since people still read horoscopes.

integrals to solve. In some sense, you can think of elliptic integrals as generalizations of inverse trig functions.

Definition 2.1. An *elliptic integral* is any function of the form

$$f(x) = \int_c^x R(t, \sqrt{P(t)})\, \mathrm{d}t,$$

where c is a constant, R is a rational function, and P is a polynomial of degree 3 or 4 with no repeated roots.

Example 2.2. When we rewrote Equation 1.1 to remove the trig functions, we obtained

$$b\int_{\sin t_0}^{\sin t_1} \frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}\, \mathrm{d}x = \int_{\sin t_0}^{\sin t_1} \frac{b\sqrt{(1-x^2)(1-k^2x^2)}}{1-x^2}\, \mathrm{d}x,$$

which is the value of an elliptic integral evaluated at $x = \sin t_1$.

Example 2.3. The elliptic genus

$$\log_\Phi(z) = \int_0^z \frac{1}{\sqrt{1-2\delta t^2+\varepsilon t^4}}\, \mathrm{d}t$$

is an elliptic integral (hence the name).

Remark 2.4. Because we watched the planets move in the sky, we eventually discovered Kepler's laws. The elliptic motion of the planets inevitably leads to the notion of elliptic integrals. How did Ochanine know to use such an elliptic integral to define a genus? Well, he didn't. Elliptic genera were defined a different way, and it is a *theorem* that their logarithm always takes the form we've seen. We'll talk about this more in a couple weeks.

Remark 2.5. Elliptic integrals were a very hot topic in the 1700s and 1800s. They were studied by the likes of Abel, Gauss, Jacobi, and Legendre, complete with all the drama. For example, Abel had been working on a result for a while, when he read a new paper of Jacobi that proved many of the things he had been working on. Abel quickly published a reply, including more results than Jacobi had released, claiming it was his "knockout of Jacobi". Gauss got word of this exchange and said to a colleague, "[Abel] has come about one third of the way that I have gone in my researches, with the same aim and even with the same choice of notation." Luckily, Abel publicly made amends for his comments shortly before his untimely passing [Gra15, §8.3].

You should take from this story three lessons:

(i) If you've been scooped, it means you're working on something interesting.

(ii) If you get a nasty referee report, you're in good company.

(iii) Make amends sooner rather than later. Life is too short to feel contentious towards anybody, let alone somebody who loves the same mathematics as you.

2.1. **First, second, and third kinds of elliptic integrals.** We've defined elliptic integrals as the integral of a certain type of function. A natural question to ask is whether there is a finite "basis" of fundamental elliptic integrals that generate the rest. The answer is yes:

Theorem 2.6. *There are three basic types of elliptic integrals, denoted $F(x;k)$, $E(x;k)$, and $\Pi(x;m,n)$, such that every elliptic integral is a linear combination of F, E, Π, and integrals of rational functions.*

Proof. The proof is pretty computational, so you can take a look at it here if you're curious: `https://mathworld.wolfram.com/EllipticIntegral.html`. This is a classical theorem, so I'm not sure who first proved it. You could almost certainly prove this yourself with enough time, patience, and motivation. □

I said that you could prove the above theorem by yourself if you really had to, but the key to this is knowing what F, E, and Π should be. You could maybe derive these yourself as well, but you would need a lot more insight. These are the *Legendre forms* of elliptic integrals. In each of these formulas, you should imagine $x = \sin\varphi$ (so $0 \le x \le 1$). We also set $0 < k^2, m, n < 1$

- Incomplete elliptic integral of the first kind:
$$F(x;k) = \int_0^x \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}\,\mathrm{d}t.$$
- Incomplete elliptic integral of the second kind:
$$E(x;k) = \int_0^x \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\,\mathrm{d}t.$$
- Incomplete elliptic integral of the third kind:
$$\Pi(x;m,n) = \int_0^x \frac{1}{(1-nt^2)\sqrt{(1-t^2)(1-mt^2)}}\,\mathrm{d}t.$$

Notice that these incomplete elliptic integrals are functions of both x and the parameters k or m and n. By evaluating at the upper bound of x, we obtain the *complete* elliptic integrals.

- Complete elliptic integral of the first kind:
$$F(k) = \int_0^1 \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}\,\mathrm{d}t.$$
- Complete elliptic integral of the second kind:
$$E(k) = \int_0^1 \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\,\mathrm{d}t.$$

- Complete elliptic integral of the third kind:

$$\Pi(m,n) = \int_0^1 \frac{1}{(1-nt^2)\sqrt{(1-t^2)(1-mt^2)}}\,\mathrm{d}t.$$

These satisfy all sorts of interesting relations and differential equations. We'll have to skip over these, since we want enough time to discuss the inverses of elliptic integrals.

3. Elliptic functions

Alright, we have some crazy new functions. We know they're important, because they came from important dynamical systems in the physical world. One of the first things you do with a new function is check whether it's invertible, and if so, try to compute the inverse.

Exercise 3.1. Prove that for any $0 < k^2, m, n < 1$, the incomplete elliptic integrals of the first, second, and third kind are invertible functions of x.

Example 3.2. Let's take a look at the incomplete elliptic integral of the first kind. Write

$$\xi(x) = \int_0^x \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}\,\mathrm{d}t$$

and denote the inverse as $\mathrm{sn}(\xi) = x$. I wanted to prove in class some amazing properties of $\mathrm{sn}(\xi)$, but ultimately decided that the sort of arguments needed are way too much computation to be enlightening on the board. For now, I'll state the amazing properties, and then I'll give you a reference if you want to see the proof.

First, $\mathrm{sn}(\xi)$ can be extended to the imaginary axis by defining $\mathrm{sn}(i\xi)$ in a suitable way. Next, there is a formula expressing $\mathrm{sn}(a+b)$ in terms of various other functions of a and b. This allows you to extend sn to a function on $\mathbb{C}$. Moreover, sn turns out to be a meromorphic function, and there are two numbers $K, K' \in \mathbb{C}$ (coming from the complete elliptic integrals of the first kind) such that $\mathrm{sn}(z+K) = \mathrm{sn}(z+K') = \mathrm{sn}(z)$ for all $z \in \mathbb{C}$.

If you repeat the story for inverting the other incomplete elliptic integrals, you will find similar properties. These inverses are called *Jacobi theta functions*.

Exercise 3.3. Using [AE06, Chapters 1 and 2] as a guide, prove that the inverses of the incomplete elliptic integrals of the first, second, and third kinds all admit meromorphic extensions with two linearly independent periods.

Question 3.4. If elliptic integrals are supposed to be generalizations of inverse trig functions, then the inverse of an elliptic integral should be a generalization of a trig function. What sort of properties might you expect of a generalized trig function?

3.1. **The general definition.** Just as with sn, all of the Jacobi theta functions can be extended to the complex plane. As functions of a complex variable, the Jacobi theta functions are meromorphic and are periodic with respect to two $\mathbb{R}$-linearly independent periods $\omega_1, \omega_2 \in \mathbb{C}$. This inspires the following definitions:

Definition 3.5. A *period* of a complex function f is a number $\omega \in \mathbb{C}$ such that $f(z+\omega) = f(z)$ for all $z \in \mathbb{C}$. A function is *doubly periodic* if there exist $\omega_1, \omega_2 \in \mathbb{C}$ with $\omega_2/\omega_1 \notin \mathbb{R}$ such that ω_1 and ω_2 are each periods of f. The pair (ω_1, ω_2) is called a *fundamental pair* of periods if every period of f is of the form $m\omega_1 + n\omega_2$ for some integers m and n.

Question 3.6. For the sake of visualization, you should think of ω_1 and ω_2 as linearly independent vectors in $\mathbb{R}^2$. Sketch a picture of a pair of periods of a doubly periodic function. What does a *fundamental pair* of periods look like?

Definition 3.7. A *period domain* of a doubly periodic function f is a parallelogram with vertices $\{0, \omega_1, \omega_2, \omega_1 + \omega_2\}$, where (ω_1, ω_2) is a fundamental pair of periods.

Definition 3.8. A function f is called *elliptic* if it is doubly periodic and meromorphic.[3]

Lemma 3.9. *If f and g are elliptic functions with the same double periodicity, then so are f', $f + g$, $f \cdot g$, and $\frac{f}{g}$.*

Proof. Each of these new functions is meromorphic (the last three cases are from standard differentiation rules; the first is from the fact that holomorphic functions are infinitely differentiable, and differentiation can only take poles to poles). Double periodicity is straightforward to check (try writing it out if you're confused). □

Theorem 3.10. *A non-constant elliptic function has a fundamental pair of periods.*

Proof. Among all periods of f, there must be a smallest one (i.e. smallest $|\omega|$). Indeed, otherwise f would have arbitrarily small non-zero periods and would thus be constant. Among all periods with smallest modulus $|\omega|$, pick one and call it ω_1. Since f has two non-colinear periods, find a period of smallest modulus in $\mathbb{C} - \mathbb{R}\{\omega_1\}$ and call it ω_2. Now by construction, there are no other periods in the triangle with vertices $\{0, \omega_1, \omega_2\}$, so we have our fundamental pair of periods. □

Remark 3.11. Every Jacobi theta function is elliptic. These come from inverting the incomplete elliptic integrals of the first, second, and third kinds, so it is tempting to speculate that the inverse of every elliptic integral is an elliptic function. However, this is not true.

Exercise 3.12. Find some examples of elliptic integrals whose inverses are not an elliptic functions. Can you find examples where the inverse is not doubly periodic? What about not meromorphic? Is it true that the inverse of an elliptic function (restricted to $\mathbb{R}$) is always an elliptic integral?

[3]Recall that a function is *meromorphic* if its only singularities in $\mathbb{C}$ are poles.

Since elliptic functions are meromorphic, they have an isolated set of singularities (if any at all). If one has any singularities, you must have infinitely many due to double periodicity. Sometimes these poles land on the boundary of a fundamental domain of the elliptic function. It is often convenient to work in regions where there are no singularities on the boundary. This leads us to the following definition:

Definition 3.13. A *cell* of an elliptic function f is a translation $D+t$ (for some $t \in \mathbb{C}$) of a fundamental domain D such that f has no poles on $\partial(D+t)$.

Question 3.14. Why does such a cell always exist?

Exercise 3.15. Prove that an elliptic function is either constant, has at least two poles in each cell, or has at least one double pole in each cell. Also prove that if an elliptic function f has no zeros in some cell, then f is constant.

The proof of the following lemma will give you a hint for Exercise 3.15.

Lemma 3.16. *The number of zeros of an elliptic function in any cell is equal to the number of poles, each counted with multiplicity.*

Proof. Let f be an elliptic function. Let C be a cell. The integral

$$\frac{1}{2\pi i}\oint_{\partial C}\frac{f'(z)}{f(z)}\,\mathrm{d}z$$

computes the difference between the number of zeros and the number of poles in C. But f' is an elliptic function with the same fundamental domain as f, so $g := f'/f$ is an elliptic function with the same fundamental domain as well. Now

$$\begin{aligned}\frac{1}{2\pi i}\oint_{\partial C} g(z)\,\mathrm{d}z &= \frac{1}{2\pi i}\left(\int_t^{t+\omega_1} + \int_{t+\omega_1}^{t+\omega_1+\omega_2} + \int_{t+\omega_1+\omega_2}^{t+\omega_2} + \int_{t+\omega_2}^{t}\right) g(z)\,\mathrm{d}z \\ &= \frac{1}{2\pi i}\int_t^{t+\omega_1}(g(z)-g(z+\omega_2))\,\mathrm{d}z - \frac{1}{2\pi i}\int_t^{t+\omega_2}(g(z)-g(z+\omega_1))\,\mathrm{d}z.\end{aligned}$$

Double periodicity implies $g(z)-g(z+\omega_1) = g(z)-g(z+\omega_2) = 0$, as desired. □

Remark 3.17. Just as elliptic integrals were an inevitable part of scientific history, so too were elliptic functions. In a few weeks, we'll see how elliptic functions very naturally lead us to modular forms and elliptic curves, two concepts at the heart of much of modern number theory. As you've seen, our story has been very analytic so far. It is extremely remarkable that algebra will play such a large role in this as well. More on this later.

Next time: Cobordism.

Daily exercises: In each lecture, I will try to give at least a couple exercises for you to think about. These may range from trivial to impossible. The point is to encourage you to think about the material outside of lecture time. I'll always put a hyperlinked list of exercises at the end of the notes to make them easy to find.

- Exercise 1.3: derive Kepler's third law.
- Exercise 1.5: write a program to numerically integrate for ellipse arc lengths.
- Exercise 3.1: prove that the incomplete elliptic integrals are invertible.
- Exercise 3.12: explore the difference between elliptic functions and inverses of elliptic integrals.
- Exercise 3.15: prove that elliptic functions are either constant, have at least two simple poles in each cell, or have at least one double pole in each cell.

References

[AE06] J. V. Armitage and W. F. Eberlein. *Elliptic functions*. Vol. 67. London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 2006, pp. xiv+387. URL: https://doi.org/10.1017/CBO9780511617867.

[Gra15] Jeremy Gray. *The real and the complex: a history of analysis in the 19th century*. Springer Undergraduate Mathematics Series. Springer, Cham, 2015, pp. xvi+350.

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