LECTURE 20: E_∞-RING SPECTRA AND DERIVED ALGEBRAIC GEOMETRY

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Next week, we're going to try to define the spectrum tmf that fits into the lifting MString $\to$ tmf of the Witten genus. Today's goal is to survey some of the algebra needed to define tmf.

1. E_∞-ring spectra

Recall that a ring spectrum is a spectrum R with a multiplication map $\mu : R \wedge R \to R$ and a unit map $u : \mathbb{S} \to R$ that together satisfy the axioms one would expect of a ring. These axioms are defined up to homotopy, since we need to work in the stable homotopy category for our smash product to be well-behaved.

It is natural to ask whether μ is commutative in some sense. It turns out that this is incredibly subtle — you might have a homotopy between μ and $\mu \circ \tau$, where τ flips $R \wedge R$, but not between the various ways to multiply $R \wedge R \wedge R$. There ends up being a lot of data that one has to keep track of, so any framework to address this question is necessarily technical. We'll discuss the oldest and lowest-tech way, which involves the notion of an *operad*.

Operads are a formalization of function composition meant to capture all of the possible associativity criteria one would need to check. We'll state the definition, which won't make any sense if you've never seen it before. Then we'll do an example, and you'll see that the definition is perfectly reasonable.

Definition 1.1. A *(symmetric) operad* consists of the data:

- a sequence of sets $\mathcal{P}(1), \mathcal{P}(2), \dots$ called *n-ary operations*,
- an element $1 \in \mathcal{P}(1)$ called the *identity*,
- for each $n \in \mathbb{N}$ and $k_1, \dots, k_n \in \mathbb{N}$, a *composition function*

$$\begin{aligned} \mathcal{P}(n) \times \mathcal{P}(k_1) \times \cdots \times \mathcal{P}(k_n) &\to \mathcal{P}(k_1 + \cdots + k_n) \\ (f, f_1, \dots, f_n) &\mapsto f \circ (f_1, \dots, f_n), \end{aligned}$$

- for each $n \in \mathbb{N}$ a right action $\cdot$ of S_n on $\mathcal{P}(n)$

satisfying the conditions:

- (identity) $f \circ (1, \dots, 1) = f = 1 \circ f$,
- (associativity)

- (equivariance).

Remark 1.2. You'll notice I didn't tell you what associativity or equivariance mean. After the next example, it will be an exercise to determine what these conditions should mean.

Example 1.3 (Little 2-disks)**.** An n-ary operation in the little 2-disks operad is a collection of n-disjoint 2-disks, labeled 1 through n. Composition is given by nesting collections of 2-disks within appropriately labeled 2-disks, erasing intermediate disks, and labeling the remaining disks appropriately. We'll draw a picture to make sense of this.

Exercise 1.4. Complete the definition of an operad by specifying what the associativity and equivariance criteria should be.

The little 2-disks operad is just one of a whole family of little n-disks operads. When $n = 1$, we get the little intervals operad. When you draw this out, you can very clearly see that the little intervals operad is encoding all possible compositions necessary to define full associativity. There can be gaps between intervals, and these correspond to the fact that we're just defining associativity up to homotopy — you can contract these gaps between your "input intervals."

As n increases, the little n-disks encode higher and higher levels of commutativity up to homotopy. There is a notion of an *algebra over an operad*, and an E_n*-algebra* is an algebra over the little n-disks operad. Rather than defining what an algebra over an operad is, we'll just state May's recognition principle:

Theorem 1.5 (May)**.** *Connected* E_n*-algebras are equivalent to n-fold loop spaces (i.e. spaces of the form* $\Omega^n X$*).*

Remark 1.6. Where does the operad structure come in? Well, if I wanted to multiply a bunch of loops $\Omega X \times \cdot \times \Omega X \to \Omega X$, I would need to concatenate my loops. All the different ways of concatenating loops correspond to the data encoded by the little intervals operad.

You may recall the Eckmann–Hilton trick that shows that $\Omega^2 X \times \Omega^2 X \to \Omega^2 X$ is commutative up to homotopy — you just shuffle the spheres around each other. The point is that if I shuffle around in some other way, I get another homotopy manifesting commutativity, but there is no relationship between these two homotopies. If I pass to $\Omega^3 X$, I get a homotopy between homotopies manifesting commutativity, and then the process stops again.

An E_∞-algebra comes with higher homotopies for all n to manifest this commutativity. This is the sort of structure one wants when speaking of commutative ring spectra. So we need to take algebras over the *little* ∞*-disks operad*, which will correspond to *infinite loops spaces*.

Definition 1.7. An E_∞-*ring spectrum* is a ring spectrum endowed with the structure of an algebra over the little ∞-disks operad.[1]

Next time: defining TMF.

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[1]Whatever all of this means. The real point to keep in mind is the story we take from loop spaces.