LECTURE 21: CONSTRUCTING TMF

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Now that we've learned the very basics about E_∞-ring spectra, we can give an overview of the construction of TMF. This will be done in the language of spectral algebraic geometry, which will be most conveniently phrased in terms of ∞-category theory. We won't be overly precise about everything, but a great place to go for technical details are Charles Rezk's notes: `https://rezk.web.illinois.edu/sag-chapter-web.pdf`. I will mostly be picking and choosing definitions and ideas from these notes to give you a sense of how TMF is defined.

1. Spectral schemes and stacks

Recall that a scheme is a (locally) ringed space: a topological space equipped with a sheaf of rings. Roughly, this sheaf of rings equips the underlying topological space with algebraic structure. Stacks are also locally ringed "spaces," although the underlying space is no longer a topological space, but rather a sheaf on a site. This indicates that as we generalize the sort of geometric objects we allow in algebraic geometry, we have to generalize what it means to be a "space."

Now that we've met a generalization of commutative rings, it's reasonable to consider locally "ringed" spaces, where we consider our sheaf to take values in a category analogous to commutative rings.

In spectral algebraic geometry, we need to generalize in both directions: instead of a sheaf of rings on a topological space, we have a sheaf of E_∞-ring spectra on an ∞-topos. It turns out that for the example we care about, our ∞-topos is more or less the same site $\mathcal{M}_{1,1}$ from ordinary algebraic geometry, so we'll only say the bare minimum about topoi.

1.1. ∞-topoi.

Our first step of abstraction was passing from topological spaces to sites. The next step of abstraction is to pass from sites to categories of sheaves on sites. Here's a rough idea of what these look like.

Definition 1.1. Given a topological space X, form its poset Open_X of open subsets. A functor $F : \mathrm{Open}_X^{\mathrm{op}} \to \mathcal{S}$ is called a *sheaf* if

$$F(U) \to \lim([n] \to \prod F(U_{i_0} \cap \cdots \cap U_{i_n}))$$

is an equivalence for all open covers $\{U_i \to U\}$ of U, for all opens U. These sheaves form a category $\mathrm{Shv}(X)$, called the *∞-topos of X*.

Definition 1.2. Given a scheme X, we can replace open subsets by the category Et_X spanned by étale morphisms $U \to X$ that factor as $U \to V \to X$, with $U \to V$ a finitely presented étale map and $V \to X$ an open affine inclusion. We can then call a functor $F : \mathrm{Et}_X^{\mathrm{op}} \to \mathcal{S}$ a *sheaf* if we get an analogous equivalence of covers.

1.2. **Spectral schemes and Deligne–Mumford stacks.** Next time, we'll talk more carefully about spectral schemes and spectral DM stacks. For now, the key things to know are that spectral schemes and stacks are given by pairs $(\mathcal{X}, \mathcal{O})$, where $\mathcal{X}$ is an ∞-topos and $\mathcal{O}$ is a sheaf of E_∞-ring spectra on $\mathcal{X}$. One can view any scheme or DM stack as a spectral scheme or DM stack, although the spectral enhancement of an ordinary scheme/stack may come with some peculiarities. The case of $\mathcal{M}_{1,1}$ is nice, in that the ∞-topos underlying the spectral enhancemet of $\mathcal{M}_{1,1}$ is 0-truncated.

1.3. **The étale site.** We begin with the definition of an étale map of rings.

Definition 1.3. A map $R \to S$ of commutative rings is *étale* if:

(i) S is finitely presented over R,

(ii) $R \to S$ is flat, and

(iii) there exists an idempotent $e \in S \otimes_R S$ such that $(S \otimes_R S)[e^{-1}] \cong S$.

Now we can talk about étale maps of E_∞-ring spectra.

Definition 1.4. A map $A \to B$ of E_∞-ring spectra is *étale* if:

(i) The induced map $\pi_0 A \to \pi_0 B$ is an étale map of commutative rings, and

(ii) $A \to B$ is *flat*, i.e. $\pi_n A \otimes_{\pi_0 A} \pi_0 B \to \pi_n B$ is an isomorphism for all n.

A finite set of étale maps $\{A_i \to A\}$ is called an *étale cover* if $\pi_0 A \to \prod \pi_0 A_i$ is faithfully flat.

Definition 1.5. Given an E_∞-ring spectrum A, sheaves on the étale site of A give us an ∞-topos $\mathrm{Shv}^{\text{ét}}(A)$. One can equip this ∞-topos with a structure sheaf of E_∞-ring spectra, yielding the *étale spectrum* $\mathrm{Spét}\, A$.

Conveniently, $\mathrm{Shv}^{\text{ét}}(A)$ is equivalent to $\mathrm{Shv}^{\text{ét}}(\pi_0 A)$, so this ∞-topos only depends on an ordinary commutative ring underlying A.

2. Topological modular forms

Recall that $(\mathcal{M}_{1,1}, \mathcal{O})$ is an étale stack. Moreover, $(\mathcal{M}_{1,1}, \mathcal{O})$ is a Deligne–Mumford stack, which means that (i) $\mathcal{M}_{1,1} \to \mathcal{M}_{1,1} \times \mathcal{M}_{1,1}$ is quasicompact, separated, and representable by algebraic spaces, and (ii) there is a surjective étale morphism $Y \to \mathcal{M}_{1,1}$ from a scheme Y. We won't need the technicalities of these statements — what matters is the following upshot: $\mathcal{M}_{1,1}$ is the colimit of all étale morphisms $\mathrm{Spét}\, A \to \mathcal{M}_{1,1}$, where A ranges over commutative rings.

Let $\mathcal{U}$ be the étale site of $\mathcal{M}_{1,1}$. The structure sheaf $\mathcal{O}$ of $\mathcal{M}_{1,1}$ can be viewed as a functor

$$\mathcal{O} : \mathcal{U}^{\mathrm{op}} \to \mathrm{CRing},$$

sending an elliptic curve $\mathrm{Spét}\, A \to \mathcal{M}_{1,1}$ to the commutative ring A. Goerrs–Hopkins–Miller showed that $\mathcal{O}$ lifts to a sheaf of E_∞-ring spectra that takes particularly interesting values.

Theorem 2.1 (Goerrs–Hopkins–Miller)**.** *There is a sheaf* $\mathcal{O}^{\mathrm{top}} \to \mathcal{U}$ *of* E_∞*-ring spectra such that the diagram*

$$\begin{array}{ccc} & & \mathrm{E}_\infty\text{−rings} \\ & \overset{\mathcal{O}^{\mathrm{top}}}{\nearrow} & \downarrow{\scriptstyle \pi_0} \\ \mathcal{U}^{\mathrm{op}} & \xrightarrow{\mathcal{O}} & \mathrm{CRing}, \end{array}$$

and such that $\mathcal{O}^{\mathrm{top}}(\mathrm{Spét}\, A \to \mathcal{M}_{1,1})$ *is the elliptic spectrum underlying the elliptic cohomology theory determined by the elliptic curve classified by* $\mathrm{Spét}\, A \to \mathcal{M}_{1,1}$.

Definition 2.2. The (periodic) spectrum of *topological modular forms* is the E_∞-ring spectrum

$$\mathrm{TMF} := \Gamma(\mathcal{M}_{1,1}, \mathcal{O}^{\mathrm{top}}) \cong \lim_{(C \to \mathrm{Spét}\, A) \in (\mathcal{M}_{1,1})_{\text{ét}}} \mathcal{O}^{\mathrm{top}}(C \to \mathrm{Spét}\, A).$$

Remark 2.3. TMF is not quite what we're looking for (in reference to a spectrum lift MString $\to$??? of the Witten genus). Next time, we'll talk about two variants of this spectrum, known as Tmf and tmf. It will turn out that the Witten genus lifts to MString $\to$ tmf, which we'll learn about after Thanksgiving break.

Next time: Tmf and tmf.

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