LECTURE 22: CONSTRUCTING TMF (CONTINUED)

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Last time, we described how to construct TMF as global sections of a sheaf of E_∞-ring spectra:

$$\mathrm{TMF} := \Gamma(\mathcal{M}_{1,1}, \mathcal{O}^{\mathrm{top}}) \cong \lim_{\mathrm{Sp\acute{e}t}\, A \to \mathcal{M}_{1,1}} \mathcal{O}^{\mathrm{top}}(\mathrm{Sp\acute{e}t}\, A \to \mathcal{M}_{1,1}).$$

Since $\mathcal{O}^{\mathrm{top}}$ is constructed so that $\mathcal{O}^{\mathrm{top}}(\mathrm{Sp\acute{e}t}\, A \to \mathcal{M}_{1,1})$ is the elliptic spectrum determined by the elliptic curve classified by $\mathrm{Sp\acute{e}t}\, A \to \mathcal{M}_{1,1}$, we find that TMF is the universal elliptic spectrum. But this was just one of our goals. The other goal was to lift the Witten genus $\Omega_*^{\mathrm{String}} \to \mathrm{MF}_*$ to the level of spectra. For this purpose, TMF is insufficient. Today, we'll work towards the desired target spectrum for lifting the Witten genus.

1. Compactification of $\mathcal{M}_{1,1}$

Over a field k, the j-invariant defines a morphism

$$j : (\mathcal{M}_{1,1})_k \to \mathbb{A}^1_k.$$

The scheme $\mathbb{A}^1_k$ is a *coarse moduli space* for $\mathcal{M}_{1,1}$ (this means that $\mathcal{M}_{1,1}$ maps uniquely to $\mathbb{A}^1_k$, and any map from $\mathcal{M}_{1,1}$ to another scheme factors through $\mathbb{A}^1_k$). It is evident that $\mathbb{A}^1_k$ is not compact, and $\mathbb{P}^1_k$ gives a natural compactification of $\mathbb{A}^1_k$.

We'd like to construct a stack $\overline{\mathcal{M}}_{1,1} \supset \mathcal{M}_{1,1}$ such that extending the j-invariant to $\mathbb{P}^1_k$ gives us a coarse moduli space

$$\bar{\mathrm{j}} : (\overline{\mathcal{M}}_{1,1})_k \to \mathbb{P}^1_k.$$

One can compute the j-invariant as $1728\frac{c_4^3}{\Delta}$, where Δ is the discriminant of the defining equation of our elliptic curve. So whatever the points of $\overline{\mathcal{M}}_{1,1} - \mathcal{M}_{1,1}$ are, they should have $j \in \{\infty\} = \mathbb{P}^1_k - \mathbb{A}^1_k$, which corresponds to $\Delta = 0$. In particular, $\overline{\mathcal{M}}_{1,1} - \mathcal{M}_{1,1}$ should consist of singular pointed cubic curves to which elliptic curves degenerate.

It turns out that we only need to add nodal cubic curves (where the base point is distinct from the node) to construct $\overline{\mathcal{M}}_{1,1}$.

Exercise 1.1. Prove that over a field k, any two nodal cubic curve with base points in the smooth loci are isomorphic. (Hint: show that normalization takes such a curve to $\mathbb{P}^1_k$ with 3 marked rational points. Then show that $\mathbb{P}^1$ marked by $\{p_1, p_2, p_3\}$ is isomorphic to $\mathbb{P}^1_k$ marked by $\{0, 1, \infty\}$.)

Recall that $\mathcal{M}_{1,1}$ was constructed as the category of elliptic curves over a base scheme, together with a forgetful functor that just returns the base scheme. We can extend this category to include nodal cubic curves. Over an arbitrary scheme, a nodal cubic curve should be a degree 3 curve $C \to S$ with an ordinary double point, together with a section $e : S \to C$ that does not meet the singular locus of C. You have to do a little work to make sense of morphisms to and from nodal cubic curves, as well as proving that this enlarged category is indeed a stack.

This is known as the *Deligne–Mumford compactification* $\overline{\mathcal{M}}_{1,1}$ of $\mathcal{M}_{1,1}$. Here, compactification refers to the fact that $|\mathcal{M}_{1,1}|$ is not a compact topological space, but $|\overline{\mathcal{M}}_{1,1}|$ is. (The vertical bars mean the points of the stack, i.e. the set of maps from the spectrum of a field into the stack, endowed with a uniquely defined but technically involved topology.) Thinking about compactifying the coarse moduli space should give you a pretty good image of what's going on here.

Remark 1.2. In the grand scheme of things, we didn't have to work too hard to compactify $\mathcal{M}_{1,1}$. Deligne–Mumford compactification and Deligne–Mumford stacks get their name from Deligne and Mumford's work on compactifying $\mathcal{M}_{g,n}$ in general. This is a much harder prospect, as well as a beautiful piece of math touching algebraic geometry, complex geometry, low-dimensional topology, and dynamics.

2. Constructing $\mathcal{O}^{\mathrm{top}}_{\overline{\mathcal{M}}_{1,1}}$

As with $\mathcal{M}_{1,1}$, we can treat $\overline{\mathcal{M}}_{1,1}$ as a spectral Deligne–Mumford stack. Last time, we used the crucial fact that Deligne–Mumford stacks (spectral or ordinary) admit affine étale covers. In practical terms, this means the étale site of $\overline{\mathcal{M}}_{1,1}$ (which consists of étale maps into $\overline{\mathcal{M}}_{1,1}$) can be computed as the colimit over the affine étale site of $\overline{\mathcal{M}}_{1,1}$. For $\mathcal{M}_{1,1}$, this allowed us to compute

$$\mathrm{TMF} := \Gamma(\mathcal{M}_{1,1}, \mathcal{O}^{\mathrm{top}}) \cong \lim \mathcal{O}^{\mathrm{top}}(\mathrm{Sp\acute{e}t}\, A \to \mathcal{M}_{1,1}).$$

Our next goal is to construct a sheaf $\mathcal{O}^{\mathrm{top}}_{\overline{\mathcal{M}}_{1,1}}$ of E_∞-ring spectra on $\overline{\mathcal{M}}_{1,1}$, after which we will define

$$\mathrm{Tmf} := \Gamma(\overline{\mathcal{M}}_{1,1}, \mathcal{O}^{\mathrm{top}}_{\overline{\mathcal{M}}_{1,1}}).$$

To simplify the notation, I'll write $\overline{\mathcal{O}}^{\mathrm{top}} := \mathcal{O}^{\mathrm{top}}_{\overline{\mathcal{M}}_{1,1}}$. The goal for the rest of the day is to give a little more detail about how $\overline{\mathcal{O}}^{\mathrm{top}}$ is constructed. The procedure is the same as the one used to construct $\mathcal{O}^{\mathrm{top}} \to \mathcal{M}_{1,1}$, the details of which we omitted last time. I'll be following Mark Behrens's fantastic notes, where the technical details are laid out very carefully: `https://math.mit.edu/~mbehrens/papers/buildTMF.pdf`. These notes make an excellent companion to Rezk's notes on viewing $\mathcal{M}_{1,1}$ or $\overline{\mathcal{M}}_{1,1}$ as a spectral DM stack (link in Lecture 21).

There are three essential tricks to constructing $\overline{\mathcal{O}}^{\mathrm{top}}$:

(i) We need to show that it suffices to define $\overline{\mathcal{O}}^{\mathrm{top}}$ on affine étale opens, much as we computed global sections as a colimit over affine étale opens.

- Given any spectral DM stack $\mathcal{X}$, the affine étale cover $i : \mathcal{X}_{\text{aff,ét}} \to \mathcal{X}_{\text{ét}}$ induces an adjoint pair of functors on the categories of presheaves (of E_∞-ring spectra):
$$i^* : \mathrm{PreShv}(\mathcal{X}_{\text{aff,ét}}) \leftrightarrows \mathrm{PreShv}(\mathcal{X}_{\text{ét}}) : i_*.$$
Here, i^* is precomposition by i, and i_* is the right Kan extension. This adjoint pair gives a Quillen equivalence of these two presheaf categories, which allows us to construct presheaves on $\mathcal{X}_{\text{ét}}$ by constructing presheaves on $\mathcal{X}_{\text{aff,ét}}$ and applying i^*. That is, it will suffice to define $\overline{\mathcal{O}}^{\text{top}}(U)$ functorially for all affine étale opens U.

(ii) We need to define $\overline{\mathcal{O}}^{\text{top}}$ rationally and after completing at each prime p. We then construct $\overline{\mathcal{O}}^{\text{top}}$ from an *arithmetic fracture square*
$$\begin{array}{ccc} \overline{\mathcal{O}}^{\text{top}} & \longrightarrow & \prod_p \overline{\mathcal{O}}_p^{\text{top}} \\ \downarrow & & \downarrow \\ \overline{\mathcal{O}}_{\mathbb{Q}}^{\text{top}} & \longrightarrow & \left(\prod_p \overline{\mathcal{O}}_p^{\text{top}}\right)_{\mathbb{Q}} \end{array}.$$

- The sheaf $\overline{\mathcal{O}}_{\mathbb{Q}}^{\text{top}}$ is easy to construct. Given an open affine $f : \operatorname{Spec} A \to \overline{\mathcal{M}}_{1,1}$, we can define an evenly graded ring $A_{2t} := \Gamma(f^*\omega^{\otimes t})$. Then $\overline{\mathcal{O}}_{\mathbb{Q}}^{\text{top}}(\operatorname{Spec} A \to \overline{\mathcal{M}}_{1,1}) = HA_*$, where $H(-)$ denotes the Eilenberg–Mac Lane spectrum. Defining the map $\overline{\mathcal{O}}^{\text{top}} \to \overline{\mathcal{O}}_{\mathbb{Q}}^{\text{top}}$ is more involved, but it only relies on making various computations in rational homotopy theory (which is worlds nicer than homotopy theory with potential torsion).

(iii) In order to define $\overline{\mathcal{O}}_p^{\text{top}}$, we have to work $K(1)$-locally, $K(2)$-locally, and use another pullback square.

- Here, $K(n)$ denotes the n^{th} Morava K-theory. This is a ring spectrum with homotopy ring $\mathbb{F}_p[v_n, v_n^{-1}]$, where $|v_n| = 2(p^n - 1)$. These play an important role in *chromatic homotopy theory*, where spectra are filtered out according to *chromatic height*. If you've never seen this before, the quick image you should have in your head is that ordinary cohomology is a height 0 spectrum, and K-theory is a height 1 spectrum. TMF will turn out to be a height 2 spectrum.

Remark 2.1. For both $\overline{\mathcal{O}}_{\mathbb{Q}}^{\text{top}}$ and the $K(n)$-localizations of $\overline{\mathcal{O}}_p^{\text{top}}$, the actual constructions involve working with formal groups, Lubin–Tate theory, and the Morava stabilizer group. This is really where the meat of Goerss–Hopkins–Miller's theorem lies, but we won't have time to get into these details.

All of this is summarized nicely by Figure 1 in Behrens's notes. This figure was generated by Aaron Mazel-Gee, who also has great notes on constructing tmf `https://etale.site/writing/tmf-seminar-talk.pdf`.

To make sense of this diagram, I need to tell you what the various stacks mean.

- $(\overline{\mathcal{M}}_{1,1})_{\mathbb{Q}} := \overline{\mathcal{M}}_{1,1} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{Q}$.
- $(\overline{\mathcal{M}}_{1,1})_p$ is the p-completion of $\overline{\mathcal{M}}_{1,1}$, i.e. the limit $\lim_n(|(\overline{\mathcal{M}}_{1,1})_{\mathbb{F}_p}|, \mathcal{O}/p^n\mathcal{O})$.
- $\overline{\mathcal{M}}_{1,1}^{\text{ord}}$ and $\overline{\mathcal{M}}_{1,1}^{\text{ss}}$ refer to *ordinary* and *supersingular* elliptic curves, respectively. An elliptic curve $E \to \operatorname{Spec} k$ in characteristic p is called *supersingular* if $E[p^n](\overline{k})$ is the trivial group for all n. If $E[p^n](\overline{k}) \cong \mathbb{Z}/p^n\mathbb{Z}$, E is said to be *ordinary*. Here, $E[p^n]$ refers to the kernel of multiplication by p^n on the group of points.

Next time: tmf and π_*tmf

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