LECTURE 23: HOMOTOPY GROUPS OF TMF

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We've met $\mathrm{TMF} := \Gamma(\mathcal{M}_{1,1}, \mathcal{O}^{\mathrm{top}})$ and $\mathrm{Tmf} := \Gamma(\overline{\mathcal{M}}_{1,1}, \overline{\mathcal{O}}^{\mathrm{top}})$. The last member of this family is tmf, which is defined as the connective cover of Tmf. You can think of this as Tmf, but with all of the negative homotopy groups removed. In practice, this is given by $\tau_{\geq 0}\mathrm{Tmf}$, where $\tau_{\geq 0}$ is the right adjoint to the inclusion of the category of spectra with trivial negative homotopy groups (i.e. *connective spectra*) into the category of all spectra.

Definition 0.1. The *(connective) spectrum of topological modular forms* is $\mathrm{tmf} := \tau_{\geq 0}\mathrm{Tmf}$.

Today's goal is to talk about the homotopy groups of TMF, Tmf, and tmf. The references we'll follow are notes by Mathew `https://math.uchicago.edu/~amathew/tmfhomotopy.pdf`, Henriques `https://people.math.rochester.edu/faculty/doug/otherpapers/TmfHomotopy.pdf`, and a paper of Bauer `https://arxiv.org/pdf/math/0311328.pdf`.

1. TMF

The first tool we have for computing $\pi_*\mathrm{TMF}$ is the *descent spectral sequence*. The basic idea is that on open affines, $\pi_j\mathcal{O}^{\mathrm{top}}$ is a sheaf of ordinary rings on $\mathcal{M}_{1,1}$. If we sheafify, we get a sheaf of ordinary rings, and we're back in the world of ordinary algebraic geometry. The descent spectral sequence has signature

$$E_2^{p,q} = H^p(\mathcal{M}_{1,1}, \pi_q\mathcal{O}^{\mathrm{top}}) \Longrightarrow \pi_{q-p}\mathrm{TMF}.$$

Let $\mathrm{MF}_* \cong \mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 - 12^3\Delta)$ be the ring of integral modular forms. Because the graded pieces of MF_* arise as global sections of $\omega^{\otimes i} \to \mathcal{M}_{1,1}$, one might expect a map $\pi_*\mathrm{TMF} \to \mathrm{MF}_*$ to pop out of the descent spectral sequence. In fact, we get a ring map

$$\pi_*\mathrm{TMF} \to \mathrm{MF}_*[\Delta^{-1}].$$

After inverting 6 (so removing 2 and 3 torsion), this is an isomorphism. In general, this map is not surjective. So this tells you that TMF only differs from modular forms at the primes 2 and 3, which are the usual suspect primes when dealing with elliptic curves. To give you a flavor of how complicated the torsion can be, let's look at $p = 3$.

Theorem 1.1. *We have*

$$\pi_*\mathrm{TMF}_{(3)} \cong \mathbb{Z}[c_4, c_6, 3\Delta, 3\Delta^2, \Delta^{\pm 3}, \alpha, \beta, b]/J,$$

where J is the ideal generated by the relations

$$\begin{aligned} c_4^3 - c_6^2 &= 576(3\Delta), \\ (3\Delta)^2 &= 3(3\Delta^2), \\ 3\alpha &= 3\beta = 3b = 0, \\ \alpha(3\Delta) &= \alpha(3\Delta^2) = \beta(3\Delta) = \beta(3\Delta^2) = 0, \\ \alpha\beta^2 &= \beta^5 = 0, \\ c_4\alpha &= c_4\beta = c_4 b = c_6\alpha = c_6\beta = c_6 b = 0. \end{aligned}$$

Rather than dwelling on $\pi_*\mathrm{TMF}$ much longer, we'll state (but not prove) a theorem that says that we might as well work with tmf instead:

Theorem 1.2. *We have* $\pi_*\mathrm{TMF} \cong \pi_*\mathrm{tmf}[\Delta^{-24}]$*. In particular,* TMF *is 576-periodic.*

Remark 1.3. The 576 comes from Δ being a weight 12 modular form, and weight k modular forms come from $\pi_{2k}\mathrm{TMF}$, so Δ^{-24} sits in degree $-24 \cdot 24 = -576$.

2. Tmf

Just as with TMF, we can define a descent spectral sequence

$$E_2^{p,q} = H^p(\overline{\mathcal{M}}_{1,1}, \pi_q \overline{\mathcal{O}}^{\mathrm{top}}) \Longrightarrow \pi_{q-p}\mathrm{Tmf}.$$

It turns out that this is much harder to analyze than the descent spectral sequence for TMF, because the cohomology of $\overline{\mathcal{M}}_{1,1}$ is more complicated.

One way to try and simplify the situation is by calculating $\pi_*(\mathrm{Tmf} \wedge X)$ for various spectra X. This doesn't need to be any easier, but clever choices of X turn out to vastly simplify the situation. At the level of the descent spectral sequence, the elliptic homology theory R_* (dual to elliptic cohomology) underlying $\operatorname{Spec} R \to \overline{\mathcal{M}}_{1,1}$ gives us a vector bundle $V \to \overline{\mathcal{M}}_{1,1}$ built out $(\operatorname{Spec} R \to \overline{\mathcal{M}}_{1,1}) \mapsto R_0(X)$. The descent spectral sequence now takes the form

$$E_2^{p,q} = H^p(\overline{\mathcal{M}}_{1,1}, V \otimes \pi_q \overline{\mathcal{O}}^{\mathrm{top}}) \Longrightarrow \pi_{q-p}(\mathrm{Tmf} \wedge X).$$

The cohomology of certain sheaves can be much simpler than that of the structure sheaf.

By definition, $\pi_n\mathrm{tmf} \cong \pi_n\mathrm{Tmf}$ for $n \geq 0$. So again, there's a close connection between the homotopy groups of the three versions of tmf, and Tmf is the most complicated of the three.

3. tmf

By construction (as global sections of a sheaf of ring spectra), tmf is a ring spectrum. Part of the data of a ring spectrum is a unit map $\mathbb{S} \to \mathrm{tmf}$, where $\mathbb{S}$ is the sphere spectrum. Taking homotopy gives me a ring morphism

$$\pi_*\mathbb{S} \to \pi_*\mathrm{tmf}$$

known as the *Hurewicz* map. The Hurewicz map is an isomorphism on $\pi_0\mathbb{S} \cong \pi_0\text{tmf} \cong \mathbb{Z}$. However, all of the torsion in $\pi_*\text{tmf}$ is 2- or 3-torsion (closely related to the fact that $\pi_*\text{TMF} \cong \text{MF}_*[\Delta^{-1}]$ after inverting 6), whereas Serre proved that $\pi_*\mathbb{S}$ is always torsion for $* > 0$. So this means that at best, $\pi_*\text{tmf}$ can only tell us about $\pi_*\mathbb{S}$ at $p = 2$ and 3.

It turns out that the Hurewicz map is not surjective even at $p = 3$. Its image is 72-periodic (because the elements Δ^3 are in the image):

$$\text{image} = \mathbb{Z}_{(3)} \oplus \alpha\mathbb{Z}/3 \oplus \bigoplus_{k\geq 0} \Delta^{3k}\{\beta, \alpha\beta, \beta^2, \beta^3, \beta^4/\alpha, \beta^4\}\mathbb{Z}/3,$$

where $|\alpha| = 3$ and $|\beta| = 10$.

Remark 3.1. Note that none of these classes lie in degree $27 + 72k$ or $75 + 72k$. There are classes of such degrees in $\pi_*\text{tmf}_{(3)}$, so the Hurewicz map is not surjective.

The 2-torsion is much more complicated — see Henriques's note (linked earlier) if you want to see some of the details.

Just as with TMF, we have a map

$$\pi_*\text{tmf} \to \text{MF}_*$$

coming from the descent spectral sequence. As before, this is an isomorphism after inverting 6, so it remains to look at the 2- and 3-torsion. We'll just close by discussing the cokernel of this map.

$$\text{coker}(\pi_n\text{tmf} \to \text{MF}_{n/2}) = \begin{cases} \mathbb{Z}/(24/\gcd(k, 24)) & n = 24k, \\ (\mathbb{Z}/2)^{\lceil (n-8)/24 \rceil} & n \equiv 4 \mod 8, \\ 0 & \text{otherwise.} \end{cases}$$

Next time: String orientation

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