LECTURE 24: STRING ORIENTATION

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For the final three lectures, we're going to work towards constructing the *string orientation*

$$\sigma : \mathrm{MString} \to \mathrm{tmf}.$$

This is an E_∞-ring spectrum map that lifts the Witten genus, in that the composite

$$\pi_{2n}\mathrm{MString} \to \pi_{2n}\mathrm{tmf} \to \mathrm{MF}_n$$

sends a $2n$-dimensional String manifold M to its Witten genus $\Phi_{\mathrm{Wit}}(M)$. But why should one expect such a lift to exist in the first place? Historically, this was actually a pleasant surprise. Before we knew that tmf had anything to do with modular forms, we calculated $H^*(\mathrm{tmf}_{(2)};\mathbb{Z}/2)$ in terms of the Steenrod algebra $\mathcal{A}$:

$$H^*(\mathrm{tmf}_{(2)};\mathbb{Z}/2) \cong \mathcal{A}/(\mathrm{Sq}^1,\mathrm{Sq}^2,\mathrm{Sq}^4).$$

This same module arises as a summand of $H^*(\mathrm{MString};\mathbb{Z}/2)$, which suggests that there should be a map of spectra $\mathrm{MString} \to \mathrm{tmf}$.

Remark 0.1. Before we start talking about some technical details needed to construct σ, here's one other reason why the string orientation is neat. Last time, we talked about the image of $\pi_{2n}\mathrm{tmf} \to \mathrm{MF}_n$. One instance where this map is not surjective is $n = 6$, in which case the image is generated by $2c_6$. So if we can construct the string orientation, and if the composite $\pi_{2n}\mathrm{MString} \to \pi_{2n}\mathrm{tmf} \to \mathrm{MF}_n$ recovers the Witten genus, then we find that the Witten genus of a 12-dimensional string manifold has even integers in its power series expansion. One can prove other similar facts, such as $\hat{\mathcal{A}}(M, TM_{\mathbb{C}}) \equiv 0 \mod 24$ when M is a 24-dimensional string manifold.

1. The complex string orientation

It turns out that the string orientation comes from trying to consider all possible E_∞-ring maps out of MString. The target of such a map will end up being an elliptic spectrum, which was another motivation for constructing tmf: the target spectrum of the lifted Witten genus should be a limit over all elliptic cohomology theories.

We'll discuss all of this in due time, but today's focus will be on a slight simplification of the story. Recall the Whitehead tower

$$\cdots \to \mathrm{BString} \to \mathrm{BSpin} \to \mathrm{BSO} \to \mathrm{BO} \to \mathrm{BO}\times\mathbb{Z},$$

which is obtained by successively killing the lowest non-trivial homotopy group. The first non-trivial homotopy group of BSpin is $\pi_7\mathrm{BSpin}$, so $\mathrm{BString} \simeq \mathrm{BO}\langle 8\rangle$. This is the source of the old notation $\mathrm{BString} = \mathrm{BO}\langle 8\rangle$ and $\mathrm{MString} = \mathrm{MO}\langle 8\rangle$.

There is a complex version of the Whitehead tower as well:

$$\cdots \mathrm{BU}\langle 6\rangle \to \mathrm{BSU} \to \mathrm{BU} \to \mathrm{BU}\times\mathbb{Z}.$$

Here, $\mathrm{BU}\langle 6\rangle$ is the fiber of $c_2 : \mathrm{BSU} \to K(\mathbb{Z},4)$. Taking the Thom spectrum of the universal bundle over $\mathrm{BU}\langle 6\rangle$ gives us the spectrum $\mathrm{MU}\langle 6\rangle$.

Recall that Quillen proved that

$$\mathbb{L} \to \mathrm{MU}$$

is an isomorphism, where $\mathbb{L}$ is the Lazard ring. In practice, this means that if E is an even periodic complex oriented spectrum, then an E_∞-ring map $\mathrm{MU} \to E$ corresponds to a formal group law on $\mathrm{Spf}(E^0(\mathbb{CP}^\infty))$.

1.1. **Multiplicative maps out of** BU**.** If we start with BU, we can look at multiplicative maps $\Sigma^\infty_+\mathrm{BU} \to E$, which correspond to ring homomorphisms $E_*\mathrm{BU} \to E_*$. Now BU has a special feature: the inclusion $\mathbb{CP}^\infty \hookrightarrow \mathrm{BU}$ implies that $E_*\mathrm{BU}$ is the symmetric algebra on $E_*\mathbb{CP}^\infty$. The upshot is that ring homomorphisms $E_*\mathrm{BU} \to E_*$ are equivalent to E_*-module maps

$$E_*\mathbb{CP}^\infty \to E_*,$$

which are elements of $E^0(\mathbb{CP}^\infty)$ by definition. And by definition of Spf, the ring $E^0(\mathbb{CP}^\infty)$ is precisely the ring of functions on $\mathrm{Spf}(E^0(\mathbb{CP}^\infty))$. But there's one technicality we've omitted: if $\Sigma^\infty_+\mathrm{BU} \to E$ is a multiplicative map, then it must preserve the units of these ring spectra. In terms of functions on $\mathrm{Spf}(E^0(\mathbb{CP}^\infty))$, this means that the base point of $\mathbb{CP}^\infty$ must be mapped to $1 \in E_*$. So if we set $G := \mathrm{Spf}(E^0(\mathbb{CP}^\infty))$, then multiplicative maps

$$\Sigma^\infty_+\mathrm{BU} \to E$$

are functions $f : G \to \pi_0 E$ satisfying $f(e) = 1$, where $e \in G$ is the unit.

1.2. **Multiplicative maps out of** BSU**.** Now we want to repeat the story for BSU. The main wrinkle is that for BU, we had a map $\mathbb{CP}^\infty \to \mathrm{BU}$ classifying $1 - L$, where L is the tautological line bundle over $\mathbb{CP}^\infty$. To lift this map to BSU, we would need $c_1(1 - L) = 0$, but this is not true. (The tautological bundle can be written as $\mathcal{O}(-1)$, from which you can calculate $c_1(1 - L) = 1$.) However, if we consider the line bundles $L_1 = L \times 1$ and $L_2 = 1 \times L$ on $\mathbb{CP}^\infty \times \mathbb{CP}^\infty$, we find that

$$\begin{aligned} c_1((1-L_1)\otimes(1-L_2)) &= c_1(1) - c_1(L_1) - c_1(L_2) + c_1(L_1\otimes L_2) \\ &= 0 - c_1(L_1) - c_1(L_2) + c_1(L_1) + c_1(L_2) \\ &= 0. \end{aligned}$$

Thus the map $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathrm{BU}$ classifying $(1 - L_1)\otimes(1 - L_2)$ lifts to a map $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathrm{BSU}$. If we repeat our discussion of multiplicative maps out of $\Sigma^\infty_+\mathrm{BU}$, we find that multiplicative maps

$$\Sigma^\infty_+\mathrm{BSU} \to E$$

correspond to functions $f \in E^0(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) = \{G \times G \to \pi_0 E\}$ satisfying

(i) $f(e, e) = 1$, where $e \in G$ is the group identity and $1 \in \pi_0 E$ is the ring unit. As before, this comes from requiring $\Sigma^\infty_+\mathrm{BSU} \to E$ to be multiplicative (in particular, to be compatible with the units associated to the ring spectrum structures).

(ii) $f(x,y) = f(y,x)$ for all $x, y \in G$. This comes from the identity $(1-L_1)\otimes(1-L_2) \cong (1-L_2)\otimes(1-L_1)$ over $\mathbb{CP}^\infty \times \mathbb{CP}^\infty$.

(iii) $f(y,z)f(x, y +_G z) = f(x,y)f(x +_G y, z)$ for all $x, y, z \in G$. This is a cocycle condition coming from the associativity of tensor products. We first compute

$$\begin{aligned}
&(1-L_1)\otimes(1-L_2)\otimes(1-L_3)\\
\cong\ &(1-L_1)\otimes(1-L_3) + (1-L_1)\otimes(1-L_2) - (1-L_1)\otimes(1-L_2\otimes L_3)\\
\cong\ &(1-L_1)\otimes(1-L_3) + (1-L_2)\otimes(1-L_3) - (1-L_1\otimes L_2)\otimes(1-L_3)
\end{aligned}$$

as bundles over $(\mathbb{CP}^\infty)^3$, where $L_1 = L \times 1 \times 1$, etc. Comparing the last two lines, we deduce

$$\begin{aligned}
&(1-L_2)\otimes(1-L_3) + (1-L_1)\otimes(1-L_2\otimes L_3)\\
\cong\ &(1-L_1)\otimes(1-L_2) + (1-L_1\otimes L_2)\otimes(1-L_3).
\end{aligned}$$

It turns out that these three criteria not only follow from $\Sigma^\infty_+\mathrm{BSU} \to E$ being multiplicative, but also characterize such maps. The upshot is that as we go up the complex Whitehead tower, multiplicative maps out of BG come with additional conditions on the functions to which they correspond. We'll discuss this in the context of $\mathrm{BU}\langle 6\rangle$ next time.

Next time: String orientation

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