LECTURE 25: STRING ORIENTATION (CONTINUED)

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Today, we will continue our discussion of the string orientation. We're in the middle of a warm-up towards the string orientation: we saw that multiplicative maps $\Sigma^\infty_+\mathrm{BU} \to E$ correspond to functions $f : \mathrm{Spf}(E^0(\mathbb{CP}^\infty)) \to \pi_0 E$ satisfying $f(e) = 1$, and multiplicative maps $\Sigma^\infty_+\mathrm{BSU} \to E$ correspond to functions $f : \mathrm{Spf}(E^0(\mathbb{CP}^\infty))^2 \to \pi_0 E$ satisfying $f(e, e) = 1$, $f(x, y) = f(y, x)$, and a cocycle condition coming from the associativity of triple tensor products of line bundles on $(\mathbb{CP}^\infty)^3$. Multiplicative maps $\Sigma^\infty_+\mathrm{BU}\langle 6\rangle \to E$ are similar. These correspond to functions $f : G^3 \to \pi_0 E$ (where $G := \mathrm{Spf}(E^0(\mathbb{CP}^\infty))$) such that:

(i) $f(e, e, e) = 1$.

(ii) $f(x_1, x_2, x_3) = f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$ for each $\sigma \in S_3$.

(iii) $f(w + x, y, z)f(w, x, z) = f(w, x + y, z)f(x, y, z) = f(w, x, y + z)f(w, y, z)$.

The last condition arises from the associativity of $\bigotimes_{i=1}^4 (1 - L_i)$, just as the cocycle condition for maps out of $\Sigma^\infty_+\mathrm{BSU}$ came from the associativity of $\bigotimes_{i=1}^3 (1 - L_i)$.

1. Cubical structures

The next goal is to recast multiplicative maps $\mathrm{MU}\langle 6\rangle \to E$ as something like functions. It turns out that functions are not quite right — we need sections of line bundles instead. Recall that the functional description of multiplicative maps $\Sigma^\infty_+\mathrm{BU} \to E$ came from recognizing $E^0(\mathbb{CP}^\infty)$ as the set of functions $G \to \pi_0 E$. Multiplicative maps $\mathrm{MU} \to E$ correspond to ring homomorphisms $\pi_*\mathrm{MU} \to \pi_* E$. How can we exploit the map $\mathbb{CP}^\infty \to \mathrm{BU}$ when we're working with the Thom spectrum? We use the Thom isomorphism to deduce that ring homomorphisms $\pi_*\mathrm{MU} \to \pi_* E$ correspond to $\tilde{E}^0(\mathbb{CP}^\infty)$. Elements of this reduced cohomology group are no longer functions on G, but rather global sections of $\mathcal{L} := \mathcal{O}(-\{e\}) \to G$.

What do criteria (i), (ii), and (iii) look like for sections of a line bundle? To answer this question, it will be most convenient to work in terms of *cubical structures*.

Given a group scheme G and a line bundle $\mathcal{L} \to G$, let $\Theta(\mathcal{L}) \to G^3$ be the line bundle whose fiber at $(x, y, z) \in G^3$ is

$$\begin{aligned}\Theta(\mathcal{L})_{(x,y,z)} &= \mathcal{L}_{x+y+z} \otimes \mathcal{L}_x \otimes \mathcal{L}_y \otimes \mathcal{L}_z \otimes \mathcal{L}^{-1}_{x+y} \otimes \mathcal{L}^{-1}_{x+z} \otimes \mathcal{L}^{-1}_{y+z} \otimes \mathcal{L}^{-1}_e \\ &= \frac{\mathcal{L}_{x+y+z}\mathcal{L}_x\mathcal{L}_y\mathcal{L}_z}{\mathcal{L}_{x+y}\mathcal{L}_{x+z}\mathcal{L}_{y+z}\mathcal{L}_e}.\end{aligned}$$

Definition 1.1. A *cubical structure* on $\mathcal{L}$ is a section $s \in \Gamma(G^3, \Theta(\mathcal{L}))$ satisfying:

(i) $s(e,e,e) = 1$.

(ii) $s(x_1,x_2,x_3) = s(x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)})$ for all $\sigma \in S_3$.

(iii) $s(w+x,y,z)\otimes s(w,x,z) = s(w,x+y,z)\otimes s(x,y,z) = s(w,x,y+z)\otimes s(w,y,z)$.

Remark 1.2. Note that the two sides of each of these equations are sections of different bundles. So in order for this to make sense, we need a *canonical* identification of the two sides of these equations. For example, $s(e,e,e)$ is a section of $\Theta(\mathcal{L})_{(e,e,e)} = \mathcal{L}_e^{\otimes 4}\otimes\mathcal{L}_e^{-\otimes 4}$, while 1 is a section of the trivial line $(\mathcal{O}_{G^3})_e$. In this case, we have a canonical isomorphism $\Theta(\mathcal{L})_{(e,e,e)} \cong \mathcal{O}_e$, which is the map sending $s(e,e,e)$ to 1 in criterion (i).

2. Theorem of the cube

Notice that the criteria in the definition of a cubical structure look an awful lot like the criteria satisfied by functions $f : G^3 \to \pi_0 E$ that correspond to multiplicative maps $\Sigma^\infty_+\mathrm{BU}\langle 6\rangle \to E$. The key differece is that cubical structures involve sections, for which we need to specify what vector space isomorphisms are playing the role of "=" in each criterion.

In terms of multiplicative maps of spectra, the transition from functions to sections comes from the Thom isomorphism. We won't be careful with the details in this class, but the punchline is that since multiplicative maps $\Sigma^\infty_+\mathrm{BU}\langle 6\rangle \to E$ correspond to elements of $E^0((\mathbb{CP}^\infty)^3)$ (which are functions on G^3), the Thom isomorphism implies that multiplicative maps $\mathrm{MU}\langle 6\rangle \to E$ correspond to elements of $\tilde{E}^0((\mathbb{CP}^\infty)^3)$ (which turn out to be sections of $\Theta(\mathcal{O}(-e))$, where global sections of $\mathcal{O}(-e)\to G$ are sections vanishing on $e : \mathrm{Spec}(\pi_0 E) \to G$). So multiplicative maps $\mathrm{MU}\langle 6\rangle \to E$ correspond to cubical structures on $\mathcal{O}(-e)$. But one thing is missing: where do the canonical trivializations for these cubical structures come from?

The answer is the theorem of the cube, a wonderful result in algebraic geometry.

Theorem 2.1 (Theorem of the cube)**.** *If G is an abelian variety and $\mathcal{L}\to G$ is a line bundle, then $\Theta(\mathcal{L})$ is the trivial line bundle.*

As stated, you might think this crucially depends on the group structure of G or the way we've defined $\Theta(\mathcal{L})$. This is exacerbated by the fact that one usually learns about the theorem of the cube in the context of abelian varieties, since this is a context where the theorem of the cube was first discovered and where many applications have been found. But in fact, the theorem of the cube is not really a theorem about group structure or even $\Theta(\mathcal{L})$. It is a result about line bundles on cartesian triples of flat proper varieties.

Theorem 2.2 (Theorem of the cube, but stronger)**.** *Let $X,Y,Z\to S$ be schemes over some scheme S. Let $x : S\to X$ and $y : S \to X$ be S-points. Let $\mathcal{L}\to X\times_S Y\times_S Z$ be a line bundle. If*

(i) $X\to S$ and $Y\to S$ are flat proper morphisms of finite presentation and with geometrically integral fibers,

(ii) $(x \times \mathrm{id}_Y \times \mathrm{id}_Z)^*\mathcal{L} \to Y \times_S Z$ *and* $(\mathrm{id}_X \times y \times \mathrm{id}_Z)^*\mathcal{L} \to X \times_S Z$ *are trivial,*

(iii) there is a point $z \in Z$ *such that* $\mathcal{L}|_{X\times Y\times\{z\}}$ *is trivial, and*

(iv) Z *is connected,*

then $\mathcal{L}$ *is trivial.*

Remark 2.3. We can deduce the first version of the theorem of the cube by noting that abelian varieties (like G) satisfy all the adjectives we're throwing at X, Y, and Z. So now you just have to check that $\Theta(\mathcal{L})$ trivializes under the desired pullbacks and restriction. You should think this through if you're still feeling uneasy about how $\Theta(\mathcal{L})$ is defined.

I highly recommend checking out the Stacks Project's treatment of the theorem of the cube: `https://stacks.math.columbia.edu/tag/0BEZ`. As is usually the case, you'll have to do a bit of a rabbit-hole-dive to get to the bottom of their proof of the theorem of the cube. But at the end, you'll get a nice picture of why this theorem is just exploiting flatness and properness to canonically trivialize certain line bundles.

Next time: String orientation

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