LECTURE 26: STRING ORIENTATION (PART 3) AND FINAL REMARKS

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Today, we reach the bittersweet end of our journey. I've learned a lot during this semester, both from preparing these lectures and from your great questions and insights in class and in office hours. Thanks for your patience with me, and for making this such a great experience! Hopefully you've learned something from the course as well.

Last time, we learned that the theorem of the cube (along with a lot of other things) implies that multiplicative maps $\mathrm{MU}\langle 6\rangle \to E$ correspond to cubical structures on $\mathcal{O}(-e) \to G$, where $G = \mathrm{Spf}(E^0(\mathbb{CP}^\infty))$ and $e : \pi_0 E \to G$ is the identity element.

As mentioned in class (but forgotten in last time's notes), G is a 1-dimensional group scheme. In order to apply the theorem of the cube, we need G to also be proper. A proper group scheme an abelian variety, and abelian varieties of dimension 1 are elliptic curves. So cubical structures don't literally correspond to all multiplicative maps $\mathrm{MU}\langle 6\rangle \to E$ for arbitrary E, since we only get the necessary trivialization data when E is an elliptic spectrum. So cubical structures on $\mathcal{O}(-e) \to G$ correspond to multiplicative maps $\mathrm{MU}\langle 6\rangle \to E$ when E is an elliptic spectrum with associated elliptic curve G. [1]

But this isn't the end of the story, since we want to understand multiplicative maps out of MString := $\mathrm{MO}\langle 8\rangle$. As before, if E is an elliptic spectrum, we get a canonical multiplicative map $\mathrm{MO}\langle 8\rangle \to E$. Moreover, these multiplicative maps are functorial in E, so that if $f : E \to E'$ is a map of elliptic spectra, then

(0.1)
$$\begin{array}{ccc} & \mathrm{MO}\langle 8\rangle & \\ \swarrow & & \searrow \\ E & \xrightarrow[f]{\hspace{3em}} & E' \end{array}$$

commutes up to homotopy. We've actually missed some spectra that admit a multiplicative map from $\mathrm{MO}\langle 8\rangle$: those whose formal group law is given by that of a nodal elliptic curve (such as KO). But if we include these as well, then the homotopy limit of diagrams (0.1) gives us a map

$$\mathrm{MO}\langle 8\rangle \to \mathrm{Tmf}.$$

Thom spectra are connective, so composing with the map $\mathrm{Tmf} \to \mathrm{tmf}$ doesn't really lose any information, and we get something looking like the string orientation

$$\mathrm{MO}\langle 8\rangle \to \mathrm{tmf}.$$

[1]Really, G is the formal neighborhood of an elliptic curve at the identity, but this is all we need for an elliptic spectrum anyway.

There are a couple questions that we need to address. First, since diagram 0.1 only commutes up to homotopy, we haven't gotten an honest map of spectra MString $\to$ tmf. Second, we haven't said anything about E_∞ structure.

This approach to the string orientation has actually never been completed, as far as I know. This was the original idea, and it led us to the right definition of Tmf as a limit over elliptic cohomology theories. But to actually construct the string orientation at the end of the day, you have to work more directly. (You still use all of the hard work about $\mathrm{MU}\langle 6\rangle$, cubical structures, and the theorem of the cube. You just don't get to use the limit of the maps $\mathrm{MO}\langle 8\rangle \to E$, which would be a harmonious way to construct the string orientation in light of the definition of Tmf.)

Remark 0.1. The passage from $\mathrm{MU}\langle 6\rangle$ to $\mathrm{MO}\langle 8\rangle$ is actually pretty subtle — it is treated thoroughly in the unpublished paper "Multiplicative orientations of KO-theory and of the spectrum of topological modular forms," by Ando, Hopkins, and Rezk. I don't think this paper is even on the arXiv, but there's a copy on Rezk's website: `https://rezk.web.illinois.edu/koandtmf.pdf`.

In this paper, the space of E_∞-ring maps $\mathrm{MO}\langle 8\rangle \to X$ is denoted by $\boldsymbol{A}(\mathrm{gl}_1 X)$. Rather than trying to work with a limit of $\boldsymbol{A}(\mathrm{gl}_1 E)$ over all elliptic spectra E, the authors work with $\boldsymbol{A}(\mathrm{gl}_1\mathrm{tmf})$. The study $\boldsymbol{A}(\mathrm{gl}_1\mathrm{tmf})$ by working rationally, completing at various primes, and working $K(1)$ or $K(2)$ locally, since tmf becomes much simpler in these settings. They then prove that $\pi_0\boldsymbol{A}(\mathrm{gl}_1\mathrm{tmf})$ is non-empty by verifying that the Eisenstein series satisfy various properties (Section 12), and checking that the E_∞-orientations MString $\to$ tmf associated to the Eisenstein series indeed lift the Witten genus (Section 15).

1. What now?

Elliptic cohomology and the Witten genus were hot topics during the late 1900s. Around the turn of the millennium, constructing and charting tmf was all the rage. To close the class, I want to gesture at a couple places (outside of chromatic homotopy) where topological modular forms are currently used in research.

- The field in which tmf plays the biggest role is quantum field theory. An important and difficult question that goes way back to the early days of elliptic cohomology is the following: what does elliptic cohomology mean geometrically? When you first learn singular cohomology or K-theory, you have explicit geometric meaning behind your cocycles. But the cocycles underlying elliptic cohomology have remained mysterious.

  Because tmf is the universal elliptic cohomology theory, this problem extends to understanding tmf cocycles. The Stolz–Teichner conjecture is a proposal that tmf cocycles correspond to certain types of 2-dimensional quantum fielld theories. (There are vast and deep generalizations of this conjecture, involving equivariance, higher heights, and so on.) The flow of information has gone both ways in this program, but one key dynamic is that tmf is a well-defined mathematical object, whereas quantum field theory still has some of the blurriness of definitions in physics. The Stolz–Teichner conjecture can be used as a tool to hone candidate definitions in physics by following mathematical definitions in around tmf.

For example, recent work of Tachikawa–Yamashita use the Stolz–Teichner conjectures and certain calculations in tmf to make predictions about quantum field theories, which they then verify directly on the physical side of the conjecture.

- In her thesis, Morgan Opie uses $\mathrm{tmf}_{(3)}$ to classify rank 3 vector bundles on $\mathbb{CP}^5$. Classifying vector bundles on projective spaces is a deceptively difficult problem. While Chern classes are good at distinguishing between vector bundles in low ranks, they become insufficient in higher ranks. Morgan constructs $\mathrm{tmf}_{(3)}$-characteristic classes, which give an extra level (or height) of structure for distinguishing bundles. I highly recommend reading her paper if you like vector bundles: `https://arxiv.org/pdf/2301.04313.pdf`.
- A great achievement of homotopy theory applied to differential topology is the (non)-uniqueness of smooth structures on spheres. Hill–Hopkins–Ravanel's solution of the Kervaire invariant one problem, together with older work of Kervaire–Milnor, implies that S^n can have a unique smooth structure only if $n = 1$, 3, 5, 13, 29, 61, and 125. It has long been known that there is a unique smooth structure for $n = 1, 3, 5$, and not a unique structure for 13 and 29. Almost 10 years ago, Wang–Xu showed that there is a unique smooth structure for 61 (via extensive Adams spectral sequence calculations involving new techniques), and that there is not a unique smooth structure for 125 by using the Hurewicz (i.e. $\pi_*\mathbb{S} \to \pi_*\mathrm{tmf}$) image in tmf. Since $\pi_*\mathrm{tmf}$ is more computable than $\pi_*\mathbb{S}$, Wang–Xu were able to make the necessary calculations in a range far beyond the limits of what we knew about $\pi_*\mathbb{S}$ at the time. (I believe we currently have a good understanding of $\pi_*\mathbb{S}$ for $* \leq 90$.)

Next time: keep me posted on all the exciting things you do in the future!

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