LECTURE 3: ELLIPTIC FUNCTIONS AND COBORDISM

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The goal today is to do some geometry and homotopy theory. But first, we need to wrap up our discussion about elliptic functions.

1. Elliptic functions

Last week, I mentioned that sn can be extended to a meromorphic function with two periods. The periodicity meshes well with the claim that elliptic integrals should be generalized inverse trig functions, since you would expect some sort of periodic behavior in their inverses (which we view as generalized trig functions).

Just as with sn, all of the Jacobi theta functions can be extended to the complex plane. As functions of a complex variable, the Jacobi theta functions are meromorphic and are periodic with respect to two $\mathbb{R}$-linearly independent periods $\omega_1, \omega_2 \in \mathbb{C}$. This inspires the following definitions:

Definition 1.1. A *period* of a complex function f is a number $\omega \in \mathbb{C}$ such that $f(z+\omega) = f(z)$ for all $z \in \mathbb{C}$. A function is *doubly periodic* if there exist $\omega_1, \omega_2 \in \mathbb{C}$ with $\omega_2/\omega_1 \notin \mathbb{R}$ such that ω_1 and ω_2 are each periods of f. The pair (ω_1, ω_2) is called a *fundamental pair* of periods if every period of f is of the form $m\omega_1 + n\omega_2$ for some integers m and n.

Question 1.2. For the sake of visualization, you should think of ω_1 and ω_2 as linearly independent vectors in $\mathbb{R}^2$. Sketch a picture of a pair of periods of a doubly periodic function. What does a *fundamental pair* of periods look like?

Definition 1.3. A *period domain* of a doubly periodic function f is a parallelogram with vertices $\{0, \omega_1, \omega_2, \omega_1 + \omega_2\}$, where (ω_1, ω_2) is a fundamental pair of periods.

Definition 1.4. A function f is called *elliptic* if it is doubly periodic and meromorphic.[1]

Lemma 1.5. *If f and g are elliptic functions with the same double periodicity, then so are f', $f + g$, $f \cdot g$, and $\frac{f}{g}$.*

Proof. Each of these new functions is meromorphic (the last three cases are from standard differentiation rules; the first is from the fact that holomorphic functions are infinitely differentiable, and differentiation can only take poles to poles). Double periodicity is straightforward to check (try writing it out if you're confused). □

Theorem 1.6. *A non-constant elliptic function has a fundamental pair of periods.*

[1] Recall that a function is *meromorphic* if its only singularities in $\mathbb{C}$ are poles.

Proof. Among all periods of f, there must be a smallest one (i.e. smallest $|\omega|$). Indeed, otherwise f would have arbitrarily small non-zero periods and would thus be constant. Among all periods with smallest modulus $|\omega|$, pick one and call it ω_1. Since f has two non-colinear periods, find a period of smallest modulus in $\mathbb{C}-\mathbb{R}\{\omega_1\}$ and call it ω_2. Now by construction, there are no other periods in the triangle with vertices $\{0,\omega_1,\omega_2\}$, so we have our fundamental pair of periods. □

Remark 1.7. Every Jacobi theta function is elliptic. These come from inverting the incomplete elliptic integrals of the first, second, and third kinds, so it is tempting to speculate that the inverse of every elliptic integral is an elliptic function. However, this is not true.

Exercise 1.8. Find some examples of elliptic integrals whose inverses are not an elliptic functions. Can you find examples where the inverse is not doubly periodic? What about not meromorphic? Is it true that the inverse of an elliptic function (restricted to $\mathbb{R}$) is always an elliptic integral?

Since elliptic functions are meromorphic, they have an isolated set of singularities (if any at all). If one has any singularities, you must have infinitely many due to double periodicity. Sometimes these poles land on the boundary of a fundamental domain of the elliptic function. It is often convenient to work in regions where there are no singularities on the boundary. This leads us to the following definition:

Definition 1.9. A *cell* of an elliptic function f is a translation $D+t$ (for some $t\in\mathbb{C}$) of a fundamental domain D such that f has no poles on $\partial(D+t)$.

Question 1.10. Why does such a cell always exist?

Exercise 1.11. Prove that an elliptic function is either constant, has at least two poles in each cell, or has at least one double pole in each cell. Also prove that if an elliptic function f has no zeros in some cell, then f is constant.

The proof of the following lemma will give you a hint for Exercise 1.11.

Lemma 1.12. *The number of zeros of an elliptic function in any cell is equal to the number of poles, each counted with multiplicity.*

Proof. Let f be an elliptic function. Let C be a cell. The integral

$$\frac{1}{2\pi i}\oint_{\partial C}\frac{f'(z)}{f(z)}\,\mathrm{d}z$$

computes the difference between the number of zeros and the number of poles in C. But f' is an elliptic function with the same fundamental domain as f, so $g:=f'/f$ is an elliptic function with the same fundamental domain as well. Now

$$\begin{aligned}\frac{1}{2\pi i}\oint_{\partial C} g(z)\ \mathrm{d}z &= \frac{1}{2\pi i}\left(\int_t^{t+\omega_1} + \int_{t+\omega_1}^{t+\omega_1+\omega_2} + \int_{t+\omega_1+\omega_2}^{t+\omega_2} + \int_{t+\omega_2}^{t}\right) g(z)\ \mathrm{d}z \\ &= \frac{1}{2\pi i}\int_t^{t+\omega_1} (g(z) - g(z+\omega_2))\ \mathrm{d}z - \frac{1}{2\pi i}\int_t^{t+\omega_2} (g(z) - g(z+\omega_1))\ \mathrm{d}z.\end{aligned}$$

Double periodicity implies $g(z) - g(z+\omega_1) = g(z) - g(z+\omega_2) = 0$, as desired. □

Remark 1.13. Just as elliptic integrals were an inevitable part of scientific history, so too were elliptic functions. In a few weeks, we'll see how elliptic functions very naturally lead us to modular forms and elliptic curves, two concepts at the heart of much of modern number theory. As you've seen, our story has been very analytic so far. It is extremely remarkable that algebra will play such a large role in this as well. More on this later.

2. Cobordism

Let's dig into cobordism. I spoke about this very informally on the first day of class. I want to do this much more rigorously today. We'll mostly focus on oriented cobordism today, where we can tell a very geometric story. We'll come back to other types of cobordism another time (whether in the next week or after about a month).

Definition 2.1. Until stated otherwise, a *manifold* is a smooth compact manifold, possibly with boundary. An *orientation* is a maximal atlas $\{(U, \varphi)\}$ such that the Jacobian of each transition function is positive: $D(\varphi \circ \psi^{-1}) > 0$.

Another way to think of this is a nowhere-vanishing n-form ω such that

$$\omega_p(\frac{\partial}{\partial x_1}|_p, \dots, \frac{\partial}{\partial x_n}|_p) > 0$$

for all p, where $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$ are induced by the atlas.

When studying cobordism with some extra structure (like oriented cobordism, complex cobordism, etc.), we need a way for the extra structure on a manifold to induce extra structure on its boundary.

Definition 2.2. An orientation on a manifold with boundary induces an orientation on the boundary as follows. The charts at the boundary are maps of the form $U \to \mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$ that reduce to charts $\mathrm{int}(U) \to \mathbb{R}^n$ on the interior. When we restrict these to the boundary, we instead get charts $\partial U \to \mathbb{R}^{n-1}$, and these together will form an oriented atlas of ∂M.

Visually, you can think of this as choosing basis vectors $v_1, \dots, v_{n-1}$ of $T_p\partial M$ (for each $p \in \partial M$) such that $v_1, \dots, v_{n-1}, v_{\text{in}}$ is a positively oriented basis of T_pM. Here, v_{in} is the *inward pointing* normal vector.

Now that we know how an orientation on a manifold W induces an orientation on its boundary ∂W, we can define *oriented cobordism*.

Definition 2.3. An *oriented cobordism* between two oriented n-manifolds M_1 and M_2 is an oriented $(n+1)$-manifold W such that $\partial W = M_1 \sqcup -M_2$, where $-M_2$ indicates that we have reversed the orientation on M_2.

Question 2.4. This is not really a question but not quite an exercise: try drawing some oriented cobordisms.

Recall that cobordism is supposed to be a particularly nice equivalence relation on the set of all manifolds. If this is true, we better be able to show that oriented cobordism is an equivalence relation.

Lemma 2.5. *Oriented cobordism is an equivalence relation.*

Proof. First, we need to show that any oriented manifold M is cobordant to itself. We can build an explicit manifold to realize this oriented cobordism: $M \times [0,1]$.

Second, we need to show that if M_1 is cobordant to M_2, then M_2 is cobordant to M_1. If W is an oriented cobordism from M_1 to M_2 (so $\partial W = M_1 \sqcup -M_2$), then note that $\partial(-W) = -M_1 \sqcup M_2$. Thus $-W$ is an oriented cobordism from M_2 to M_1.

Finally, we need to show that if M_1 is cobordant to M_2 (via W), and if M_2 is cobordant to M_3 (via V), then we can glue W and V along the boundary components $-M_2$ and M_2 to get a new manifold Y. The key to making this work is having the orientation pointing out of W and into V. By construction, $\partial Y = M_1 \sqcup -M_3$. □

Remark 2.6. Note that if $\partial M \neq \varnothing$, then M cannot be cobordant to any other manifold. Indeed, such a cobordism would be a manifold W with M a component of ∂W, so ∂M would be a component of $\partial\partial W$. Now what do you know about the boundary of a boundary?

After quotienting the set of all manifolds *without boundary* by this equivalence relation, we get a set of oriented cobordism classes. The next step is to give this set the structure of a group.

Lemma 2.7. *Disjoint union turns the set of oriented cobordism classes into a group.*

Proof. The group identity is given by $\varnothing$, which we think of as a manifold of any dimension. The group inverse of M is $-M$: we get an oriented cobordism from $M \sqcup -M$ to $\varnothing$ by bending the cylinder $M \times [0,1]$. This cylinder has boundary $M \sqcup -M$, which is how we saw that oriented cobordism is reflexive just a moment ago. But if we instead think of this boundary as $(M \sqcup -M) \sqcup (\varnothing)$, we now see that $M \sqcup -M$ is cobordant to $\varnothing$.

We're not done yet! We also have to check that this group operation is well-defined. If M_1 is cobordant to M_2 via W, and if M_1' is cobordant to M_2' via W', then we need to see that $M_1 \sqcup M_1'$ is cobordant to $M_2 \sqcup M_2'$. The desired cobordism is given by $W \sqcup W'$. □

Recall also that we can put a ring structure on Ω_*^{SO}, as long as we do not restrict to any one dimension.

Lemma 2.8. *Cartesian product turns the group of oriented cobordism classes into a graded ring.*

Proof. To see that the structure is graded, note that $\dim(M_1 \times M_2) = \dim(M_1) \cdot \dim(M_2)$. To see that this product is well-defined in cobordism, we need to show that if M_1 is cobordant to M_2 and M_1' is cobordant to M_2', then $M_1 \times M_1'$ is cobordant to $M_2 \times M_2'$. Finally, we need to check the axioms of the ring structure. We leave these last two steps as an exercise. □

Exercise 2.9. Check that oriented cobordism with Cartesian product and disjoint union satisfies the axioms of a ring structure, and that the product is well-defined.

2.1. **Low-dimensional examples.** Let's compute the group Ω_n^{SO} by hand for $0 \leq n \leq 2$. This will boil down to the fact that we can classify all oriented manifolds (without boundary) in these dimensions.

First, let's look at Ω_0^{SO}. A 0-dimensional manifold is a finite union of points. An orientation on a point is just an assignment of ± 1; denote a positive point by pt_+, and a negative point by pt_-. We have cobordisms from pt_+ to pt_- and pt_- to pt_+ given by $[0,1]$ and $-[0,1]$, respectively. In particular, $-\mathrm{pt}_+ = \mathrm{pt}_-$.

Proposition 2.10. *The function $f : \mathbb{Z} \to \Omega_0^{\mathrm{SO}}$ given by $f(n) = \coprod_n \mathrm{pt}_+$ is a group isomorphism.*

Proof. We just need to show that any oriented 0-manifold is cobordant to $\varnothing$, $\coprod_n \mathrm{pt}_+$, or $\coprod_n \mathrm{pt}_-$ for some n. It will then follow from the previous paragraph that f is a bijective group homomorphism, as desired.

In general, a 0-dimensional oriented manifold takes the form $\coprod_a \mathrm{pt}_+ \sqcup \coprod_b \mathrm{pt}_-$. If $a \geq b$, then we can pair off positive and negative points (since $-\mathrm{pt}_+ = \mathrm{pt}_-$) to obtain an oriented cobordism to $\coprod_{a-b} \mathrm{pt}_+$. If $a \leq b$, then we instead get a cobordism to $\coprod_{b-a} \mathrm{pt}_-$. In any case, we find that any 0-dimensional oriented manifold is cobordant to $\varnothing$ or a collection of points with the same orientation. □

Example 2.11. Every 1-dimensional manifold (without boundary) is a finite disjoint union of circles, each with an orientation. Note that S^1 bounds the oriented disk D^2, so every circle is cobordant to $\varnothing$. Thus each component of any 1-dimensional manifold is cobordant to $\varnothing$, so $\Omega_1^{\mathrm{SO}} = 0$.

Example 2.12. By the classification of orientable surfaces, connected orientable surfaces (without boundary) are determined by their genus. The genus 0 surface is S^2, which bounds the oriented 3-ball D^3. The genus 1 surface is the torus T^2, which bounds the oriented handlebody $D^2 \times S^1$. The genus g surface Σ_g is the connect sum $\#_g T^2$ and bounds the oriented handlebody $\#_g(D^2 \times T^1)$. It follows that every orientable surface is a disjoint union of components that are cobordant to $\varnothing$, so $\Omega_2^{\mathrm{SO}} = 0$.

Remark 2.13. One can also prove that $\Omega_3^{\mathrm{SO}} = 0$ in a fairly direct way. This would require a bit of a diversion into handlebody decompositions of 3-folds, as well as some surgery theory. We'll have to skip this for brevity's sake, but I can round up some references if anybody is interested.

Remark 2.14. Next time, we'll use an analytic construction called the *signature* of a $4n$-manifold to prove that Ω_4^{SO} is not trivial. This will be our natural segue into genera. Today, we'll see a homotopical proof of this fact (and much more).

2.2. Thom spaces, Thom spectra, and the Pontryagin–Thom construction.

On the first day of class, I mentioned that $\Omega_n^{\mathrm{SO}} \otimes \mathbb{Q} \cong \mathbb{Q}[\mathbb{CP}^2, \mathbb{CP}^4, \ldots]$. Based on what we've done so far, you should be utterly amazed by this sort of theorem. How can one possibly get at all of this topology in every dimension? What would a proof even look like?

Well, this is the perfect opportunity to introduce the magic of René Thom. The motto is:

> *Turn a question about the geometry or topology of an object X into a question about the homotopy theory of an object Y.*

Remark 2.15. This motto has been shockingly effective over the last 70 years. In recent years, arithmetic questions have also been successfully attacked using homotopy theory. A reason for all of this is that homotopy theory is an excellent organizational tool, and there are lots of powerful computational tools present in the subject.

As some basic language, we need to introduce the notion of a spectrum. First, recall the definition of a *loop space*:

Definition 2.16. A *pointed space* is a pair (X, x), where X is a topological space and $x \in X$ is a chosen base point. The *loop space* $\Omega_x X$ (or just ΩX) of (X, x) is the topological space of continuous maps $[(S^1, 0), (X, x)]$, where $0 \in S^1$ is a chosen base point. Here, we topologize the set $[(S^1, 0), (X, x)]$ via the compact-open topology.

Now we can introduce *spectra*. We'll do these in more detail later, so take this as a first approximation of a richer story.

Definition 2.17. A *spectrum* E is a sequence of pointed topological spaces $\{E_n\}_{n\geq 0}$, together with *structure maps* $e_n : E_n \to \Omega E_{n+1}$.

A morphism of spectra $\varphi : E \to F$ is a sequence of maps of pointed topological spaces $\varphi_n : E_n \to F_n$ such that the following diagrams all commute:

$$\begin{array}{ccc} E_n & \xrightarrow{\varphi_n} & F_n \\ \downarrow{\scriptstyle e_n} & & \downarrow{\scriptstyle f_n} \\ \Omega E_{n+1} & \xrightarrow{\Omega\varphi_{n+1}} & \Omega F_{n+1}. \end{array}$$

Remark 2.18. Recall that a map of topological spaces is called a *weak equivalence* if it induces an isomorphism on homotopy groups. We have the same notion for spectra, where we replace homotopy groups with *stable* homotopy groups

$$\pi_n^s E := \operatorname*{colim}_{k\to\infty} \pi_k E_{k+n}.$$

Remark 2.19. What is going on here? There's a theorem known as *Brown representability*, which very roughly says that cohomology should be representable by a sequence of spaces. That is, there should be spaces $\{E_n\}$ such that $H^n(X) = [X, E_n]$. But cohomology isn't just a bucket of groups — there should be some connection between H^n and H^{n+1}. Spectra naturally come out of this story, and the structure maps tie all the different dimensions together. We'll talk about this more rigorously later.

Now that I've given you a definition, we need to see an example. I anticipate that we'll be out of time by this point (if not earlier), so I'll just give you a preview. Associated to the classifying space $BG(n)$ of some Lie group $G(n)$, we will build a topological space known as the *Thom space*. This will be the one-point compactification $\mathrm{Th}(\xi_n) := \xi_n \cup \{\infty\}$ of the universal bundle $\xi_n \to BG(n)$. We will see that this construction comes with very natural structure maps $\mathrm{Th}(\xi_n) \to \Omega\mathrm{Th}(\xi_{n+1})$, so that the Thom spaces fit together to form the *Thom spectrum* MG. Finally, we will talk about the *Pontryagin–Thom isomorphism*, which relates the homotopy groups of the spectrum MG to the cobordism ring Ω_*^G.

Next time: More cobordism, then genera.

Daily exercises: In each lecture, I will try to give at least a couple exercises for you to think about. These may range from trivial to impossible. The point is to encourage you to think about the material outside of lecture time. I'll always put a hyperlinked list of exercises at the end of the notes to make them easy to find.

- Exercise 1.8: explore the difference between elliptic functions and inverses of elliptic integrals.
- Exercise 1.11: prove that elliptic functions are either constant, have at least two simple poles in each cell, or have at least one double pole in each cell.
- Exercise 2.9: check the ring structure on Ω_*^{SO}.

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