LECTURE 6: PONTRYAGIN–THOM (CONTINUED)

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We ended last time in the middle of the Pontryagin–Thom construction. Let me remind you where we were with that.

Definition 0.1. The Thom spectrum MSO is defined as the spectrum with spaces $\mathrm{Th}(\xi_n)$, where ξ_n is the universal bundle on $B\mathrm{SO}(n)$. The structure maps are given by $\Sigma\mathrm{Th}(\xi_n) \to \mathrm{Th}(\xi_{n+1})$, which are equivalent to maps $\mathrm{Th}(\xi_n) \to \Omega\mathrm{Th}(\xi_{n+1})$ under the loops-suspension adjunction.

1. Pontryagin–Thom isomorphism

We can now state the big theorem.

Theorem 1.1 ((Pontryagin–)Thom)**.** *There is a ring isomorphism* $\Omega_*^{\mathrm{SO}} \cong \pi_*\mathrm{MSO}$.

To prove this theorem, we need to do the following:

(i) Construct functions $f_n : \Omega_n^{\mathrm{SO}} \to \pi_n\mathrm{MSO}$ for all $n \geq 0$.

(ii) Construct functions $g_n : \pi_n\mathrm{MSO} \to \Omega_n^{\mathrm{SO}}$ for all $n \geq 0$.

(iii) Prove that $(f_n)^{-1} = g_n$ for all $n \geq 0$.

(iv) Prove that f_* and g_* are ring homomorphisms.

We will do steps (i) and (ii). Step (iii) is a matter of working through the constructions that show up in (i) and (ii) to check that f_n and g_n are mutually inverse. Step (iv) then boils down to checking that f_* and g_* are additive over disjoint unions, factor over Cartesian products, and preserve the multiplicative identity.

Exercise 1.2. Verify steps (iii) and (iv).

1.1. The Pontryagin–Thom construction.

We'll first construct $f_n : \Omega_n^{\mathrm{SO}} \to \pi_n\mathrm{MSO}$ via the *Pontryagin–Thom construction.* Given an oriented n-manifold M, use the Whitney embedding theorem to embed M in $\mathbb{R}^{n+k}$ for some k. The *normal bundle* N_M of $i : M \hookrightarrow \mathbb{R}^{n+k}$ is the quotient $\mathbb{R}^{n+k}/TM$, which is a vector bundle of rank k. The *tubular neighborhood theorem* gives us an embedding $j : N_M \hookrightarrow \mathbb{R}^{n+k}$ such that the zero section of j is the embedding $i : M \to \mathbb{R}^{n+k}$.

Note that if we collapse all of $\mathbb{R}^{n+k} - j(N_M)$ to a point, we are taking a one point compactification of N_M. In other words, this gives us the Thom space $\mathrm{Th}(N_M)$. We can extend this map to $\mathbb{R}^{n+k} \cup \{\infty\}$ by sending ∞ to the compactifying point as well. Altogether, we have a composite map

$$S^{n+k} = \mathbb{R}^{n+k} \cup \{\infty\} \to \mathbb{R}^{n+k}/(\mathbb{R}^{n+k} - j(N_M)) = \mathrm{Th}(N_M).$$

Finally, since $N_M \to M$ is a rank k vector bundle, it arises as the pullback of $\xi_k \to B\mathrm{SO}(k)$ under map $p : M \to B\mathrm{SO}(k)$. This setup induces a map $\mathrm{Th}(N_M) \to \mathrm{Th}(\xi_k)$. Note that the homotopy class of the composite

$$S^{n+k} = \mathbb{R}^{n+k} \cup \{\infty\} \to \mathbb{R}^{n+k}/(\mathbb{R}^{n+k} - j(N_M)) = \mathrm{Th}(N_M) \to \mathrm{Th}(\xi_k)$$

is an element of $\pi_{n+k}(\mathrm{MSO}(k))$, which in turn gives us an element of $\pi_n\mathrm{MSO}$. Define $f_n(M) \in \pi_n\mathrm{MSO}$ to be this element.

Lemma 1.3. *The Pontryagin–Thom construction $f_n : \Omega_n^{\mathrm{SO}} \to \pi_n\mathrm{MSO}$ is well-defined.*

Proof. We made three choices along the way: we chose an embedding, a tubular neighborhood, and a classifying map of the normal bundle. Any two tubular neighborhoods are isotopic, and any two classifying maps are homotopic, so neither of these choices changes the homotopy class of our composite map. To show that f_n is independent of our choice of embedding, we'll just wave our hands and say that you can take the standard embedding of $\mathbb{R}^{n+k}$ into $\mathbb{R}^{n+k+1}$, apply the construction there, and show that the resulting element of $\pi_{n+k+1}(\mathrm{MSO}(k+1))$ yields the same element of $\pi_n\mathrm{MSO}$. We can thus assume that $k > n+1$, and then show that any two embeddings into $\mathbb{R}^{n+k}$ must be isotopic.

Finally, we need to show that if M_1 and M_2 are cobordant, then $f_n(M_1) = f_n(M_2)$. We'll have to wave our hands at this for time's sake and say that applying the Pontryagin–Thom construction to the cobordism W will give us a homotopy between the composite maps $f_n(M_1)$ and $f_n(M_2)$. □

1.2. **Transversality.** Now we will use transversality to construct $g_n : \pi_n\mathrm{MSO} \to \Omega_n^{\mathrm{SO}}$. An element α of $\pi_n\mathrm{MSO}$ is represented by a map $S^{n+k} \to \mathrm{MSO}(k)$ for some $k \geq n+1$. If this were a map of finite dimensional manifolds, we could try using the implicit function theorem to construct and element of Ω_n^{SO} (which is what we'll eventually do). But for now, we don't really know what $\mathrm{MSO}(k)$ actually looks like. It turns out that $B\mathrm{SO}(k)$ is the *infinite real Grassmannian of oriented k-planes*. The oriented Grassmannian $\mathrm{Gr}_k(\mathbb{R}^\ell)$ is a manifold parameterizing oriented k-planes in $\mathbb{R}^\ell$ (which you can realize as the oriented double cover of the usual Grassmannian), and $B\mathrm{SO}(k)$ is the colimit $\operatorname{colim}_{\ell\to\infty} \mathrm{Gr}_k(\mathbb{R}^\ell)$. Each of these oriented Grassmannians comes with a canonical bundle $\xi_{k,\ell}$, and $\mathrm{MSO}(k)$ is given by $\operatorname{colim}_{\ell\to\infty} \mathrm{Th}(\xi_{k,\ell})$.

All of this tells us that $S^{n+k} \to \mathrm{MSO}(k)$ factors through $\gamma : S^{n+k} \to \mathrm{Th}(\xi_{k,\ell})$ for some ℓ, where $\xi_{k,\ell}$ is the canonical bundle of the Grassmannian of oriented k-planes in $\mathbb{R}^\ell$. Since $\mathrm{Th}(\xi_{k,\ell})$ is a Thom space over a compact base, $\mathrm{Th}(\xi_{k,\ell}) - \{\infty\}$ is homeomorphic to the total space of $\xi_{k,\ell}$ (by definition). Let $U = S^{n+k} - \gamma^{-1}(\infty) \subseteq \mathbb{R}^{n+k}$. We thus have a continuous map $\gamma|_U : U \to \xi_{k,\ell}$ of smooth manifolds, so $\gamma|_U$ is homotopic to a smooth map. Smooth maps can always be perturbed to be transverse to a given submanifold on the target, so we may assume that γ is transverse to the zero section in $\xi_{k,\ell}$.

The total space $\xi_{k,\ell}$ has dimension $k + \dim \mathrm{Gr}_k(\mathbb{R}^\ell)$, and the zero section has dimension $\mathrm{Gr}_k(\mathbb{R}^\ell)$. In other words, the zero section has codimension k, and $\gamma : U \to \xi_{k,]ell}$ is a smooth map that is transverse to this zero section. Now by the inverse image theorem, the inverse image of the zero section under γ is a smooth, oriented manifold $M \subset \mathbb{R}^{n+k}$ of codimension k. Define $g_n(\alpha) := M$.

Lemma 1.4. *The map $g_n : \pi_n\mathrm{MSO} \to \Omega_n^{\mathrm{SO}}$ is well-defined.*

Proof. For the sake of time, we'll say even less about this than the previous lemma. We first need to show that g_n is independent of the choice of representative $S^{n+k} \to \mathrm{MSO}(k)$, as well as the perturbations we used to get transversality.

Increasing k changes the dimension of the ambient space in which M is embedded, but does not change M itself.

If $\alpha_1, \alpha_2 : S^{n+k} \to \mathrm{MSO}(k)$ are two homotopy equivalent maps, then this homotopy will yield a cobordism from $g_n(\alpha_1)$ to $g_n(\alpha_2)$. This means that choosing a different representative of α or perturbing α to get smoothness and transversality will not change the cobordism class of M. □

2. Computing $\pi_*\mathrm{MSO} \otimes \mathbb{Q}$

Theorem 1.1 tells us that if we want to compute Ω_*^{SO}, it suffices to compute $\pi_*\mathrm{MSO}$. How hard could it be? Well it turns out that this is doable, but it's still pretty hard. If you want to know the answer, you'll have to go to the great work of Wall [Wal60].

We'll just answer the easier question of computing $\pi_*\mathrm{MSO}\otimes\mathbb{Q}$. This will be a compilation of a few lemmas, each with some content that we won't entirely prove. Before we begin, we need to define the homology of the spectrum MSO.

Definition 2.1. The *stable rational homology groups* of MSO are defined as

$$H_n(\mathrm{MSO}; \mathbb{Q}) := \operatorname*{colim}_{k\to\infty} H_{n+k}(\mathrm{MSO}(k); \mathbb{Q}).$$

Lemma 2.2. *We have $\pi_*\mathrm{MSO} \otimes \mathbb{Q} \cong H_*(\mathrm{MSO}; \mathbb{Q})$.*

Proof. This is an isomorphism between homotopy groups and homology groups — if you've taken algebraic topology before, your brain should be screaming, "Hurewicz!" We only need the rational version of the Hurewicz theorem, which says that if X is simply connected with $\pi_i(X) \otimes \mathbb{Q} = 0$ for $i \leq r$, then $\pi_i(X) \otimes \mathbb{Q} \to \tilde{H}_i(X; \mathbb{Q})$ is an isomorphism for $0 \leq i \leq 2r$ and a surjection for $i = 2r + 1$.

Now let's think about what $\pi_*\mathrm{MSO}$ and $H_*(\mathrm{MSO})$ mean. We want to relate

$$\operatorname{colim} \pi_{n+k}\mathrm{MSO}(k) \otimes \mathbb{Q} \quad \text{and} \quad \operatorname{colim} H_{n+k}(\mathrm{MSO}(k); \mathbb{Q}).$$

I will leave it as an exercise that $\mathrm{MSO}(k)$ is simply connected with $\pi_i(\mathrm{MSO}(k)) = 0$ for $1 \leq i \leq k-1$. The rational Hurewicz theorem now tells us that $\pi_{i+k}\mathrm{MSO}(k) \otimes \mathbb{Q} \cong H_{i+k}(\mathrm{MSO}(k); \mathbb{Q})$ for $0 \leq i \leq 2k-2$. Taking the colimit as $k \to \infty$ gives us the desired result. □

Exercise 2.3. Prove that $\mathrm{MSO}(k)$ is simply connected with $\pi_i(\mathrm{MSO}(k)) = 0$ for $1 \leq i \leq k-1$. Remember that $\mathrm{MSO}(k)$ is a Thom space of a rank k vector bundle over $B\mathrm{SO}(k)$.

Lemma 2.4. *We have* $H_*(\mathrm{MSO}; \mathbb{Q}) \cong H_*(B\mathrm{SO}; \mathbb{Q})$.

Proof. This is an application of the Thom isomorphism from earlier. We had phrased it in terms of cohomology, but on homology we get an isomorphism $H_{i+k}(\mathrm{MSO}(k); \mathbb{Q}) \cong H_i(B\mathrm{SO}(k); \mathbb{Q})$. Here, the degree k shift comes from the fact that $\mathrm{MSO}(k)$ is the Thom space of a rank k vector bundle over $B\mathrm{SO}(k)$.

It remains to show that $H_i(B\mathrm{SO}; \mathbb{Q}) \cong \operatorname{colim} H_i(B\mathrm{SO}(k); \mathbb{Q})$. To see this, it suffices to show that the inclusion $B\mathrm{SO}(k) \to B\mathrm{SO}$ is k-*connected*. It follows that $H_i(B\mathrm{SO}(k); \mathbb{Q}) \cong H_i(B\mathrm{SO}; \mathbb{Q})$ for $k > i$, so the desired result holds by taking the colimit as $k \to \infty$. □

Exercise 2.5. Show that the inclusion $B\mathrm{SO}(k) \to B\mathrm{SO}$ is k-connected. Recall that a continuous map $f : X \to Y$ of topological spaces is k-*connected* if for all $x \in X$, the induced maps $f_* : \pi_i(X, x) \to \pi_i(Y, f(x))$ are isomorphisms for all $i < k$ and a surjection for $i = k$.

Remark 2.6. So far, we've turned a geometric question (computing $\Omega_*^{\mathrm{SO}} \otimes \mathbb{Q}$) into a homotopical question (computing $\pi_*\mathrm{MSO} \otimes \mathbb{Q}$), which we've now turned into an algebraic question (computing $H_*(B\mathrm{SO}; \mathbb{Q})$).

We had to do some math for each of these translations, but it feels like everything we've done so far is easier than computing Ω_*^{SO} itself. By *conservation of math*, there should be as much mathematical content (whatever that means) in proving Ω_*^{SO} geometrically or via our current route. The next step is to compute $H_*(B\mathrm{SO}; \mathbb{Q})$. This is a "standard" computation, but it generally involves several other theorems.

Lemma 2.7. *We have* $H_*(B\mathrm{SO}; \mathbb{Q}) \cong \mathbb{Q}[p_1, p_2, \ldots]$, *where* $|p_i| = 4i$.

Proof. An H-*space* is the homotopy-theoretic analog of a topological group: it is a topological space X with a chosen element $e \in X$, a continuous map $\mu : X \times X \to X$, and for each $x \in X$ a homotopy from $x \mapsto \mu(x, e)$ to the identity map and $x \mapsto \mu(e, x)$ to the identity map. H-spaces have lots of nice properties. For example, the Milnor–Moore theorem states that if X is a path connected H-space, then $H_*(X; \mathbb{Q})$ is a free graded-commutative algebra on $\pi_*(X) \otimes \mathbb{Q}$.

It turns out that $B\mathrm{SO}$ is an H-space (see Exercise 2.8). So if we want to compute $H_*(B\mathrm{SO}; \mathbb{Q})$, all we have to do is compute $\pi_*(B\mathrm{SO}) \otimes \mathbb{Q}$. Well, using the loop-suspension adjunction, we find that

$$\begin{aligned}[S^i, B\mathrm{SO}(n)] &\cong [\Sigma S^{i-1}, B\mathrm{SO}(n)] \\ &\cong [S^{i-1}, \Omega B\mathrm{SO}(n)] \\ &\cong [S^{i-1}, \mathrm{SO}(n)].\end{aligned}$$

In particular, $\pi_i(B\mathrm{SO}(n)) \cong \pi_{i-1}(\mathrm{SO}(n))$. Now $\mathrm{SO}(n)$ is the connected component of $\mathrm{O}(n)$, so $\pi_i(\mathrm{SO}(n)) = \pi_i(\mathrm{O}(n))$ for all $i \geq 1$. The homotopy groups of $\mathrm{O} = \operatorname{colim} \mathrm{O}(n)$ are famously given by the Bott periodicity theorem:

$n \pmod 8$	0	1	2	3	4	5	6	7
$\pi_n \mathrm{O}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

Altogether, we find that $\pi_*(B\mathrm{SO})$ is $\mathbb{Z}/2$ when $* \equiv 1$ or 2 mod 8, $\mathbb{Z}$ when $* \equiv 0$ mod 4, and 0 otherwise. Tensoring with $\mathbb{Q}$ kills the torsion groups, so we find that $\pi_*(B\mathrm{SO}) \otimes \mathbb{Q}$ is rank 1 in dimensions $4n > 0$. □

Exercise 2.8. Prove that direct sums of principal SO-bundles gives $B\mathrm{SO}$ the structure of an H-space.

The generators p_i are typically chosen to be the *Pontryagin classes*, which belong in any basic toolkit of cohomology classes.

We're almost there. The last thing we need to do is explain why $H_*(B\mathrm{SO}; \mathbb{Q})$ can be generated by $\{\mathbb{CP}^2, \mathbb{CP}^4, \ldots\}$ instead of $\{p_1, p_2, \ldots\}$.

Exercise 2.9. Building on last lecture, show that the signature of a $4n$-manifold induces a group homomorphism $\sigma : \Omega_{4n}^{\mathrm{SO}} \to \mathbb{Z}$. Prove that $\sigma(\mathbb{CP}^{2n}) = 1$, and deduce that $\{\mathbb{CP}^2, \mathbb{CP}^4, \ldots\}$ generate $\Omega_*^{\mathrm{SO}} \otimes \mathbb{Q}$.

Next time: Hirzebruch signature theorem, then modular formss. We'll come back to genera later.

Daily exercises: I decided to stop collecting the exercises here. If you really want me to put them at the end of the notes like before, let me know!

References

[Wal60] C. T. C. Wall. "Determination of the cobordism ring". In: *Ann. of Math. (2)* 72 (1960), pp. 292–311. URL: https://doi.org/10.2307/1970136.

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