LECTURE 7: HIRZEBRUCH SIGNATURE THEOREM AND THE WEIERSTRASS $\wp$-FUNCTION

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Today, we'll wrap up our discussion about genera for a while, and then hopefully switch gears to modular forms. Last time, we finished proving that $\Omega_*^{\mathrm{SO}} \otimes \mathbb{Q} \cong \mathbb{Q}[p_1, \ldots] \cong \mathbb{Q}[\mathbb{CP}^2, \ldots]$. We have a few loose ends: what are these Pontryagin classes doing here, and what can we say about the "signature" genus? We'll answer both of these questions, and we'll hopefully gain an appreciation for the characteristic series of a genus along the way.

1. Computing $\pi_*\mathrm{MSO} \otimes \mathbb{Q}$

Last time, we proved that $\pi_*\mathrm{MSO} \otimes \mathbb{Q} \cong H_*(B\mathrm{SO}; \mathbb{Q})$. Let's finish off the calculation.

Lemma 1.1. *We have $H_*(B\mathrm{SO}; \mathbb{Q}) \cong \mathbb{Q}[p_1, p_2, \ldots]$, where $|p_i| = 4i$.*

Proof. An *H-space* is the homotopy-theoretic analog of a topological group: it is a topological space X with a chosen element $e \in X$, a continuous map $\mu : X \times X \to X$, and for each $x \in X$ a homotopy from $x \mapsto \mu(x, e)$ to the identity map and $x \mapsto \mu(e, x)$ to the identity map. H-spaces have lots of nice properties. For example, the Milnor–Moore theorem states that if X is a path connected H-space, then $H_*(X; \mathbb{Q})$ is a free graded-commutative algebra on $\pi_*(X) \otimes \mathbb{Q}$.

It turns out that $B\mathrm{SO}$ is an H-space (see Exercise 1.2). So if we want to compute $H_*(B\mathrm{SO}; \mathbb{Q})$, all we have to do is compute $\pi_*(B\mathrm{SO}) \otimes \mathbb{Q}$. Well, using the loop-suspension adjunction, we find that

$$\begin{aligned}[S^i, B\mathrm{SO}(n)] &\cong [\Sigma S^{i-1}, B\mathrm{SO}(n)] \\ &\cong [S^{i-1}, \Omega B\mathrm{SO}(n)] \\ &\cong [S^{i-1}, \mathrm{SO}(n)].\end{aligned}$$

In particular, $\pi_i(B\mathrm{SO}(n)) \cong \pi_{i-1}(\mathrm{SO}(n))$. Now $\mathrm{SO}(n)$ is the connected component of $\mathrm{O}(n)$, so $\pi_i(\mathrm{SO}(n)) = \pi_i(\mathrm{O}(n))$ for all $i \geq 1$. The homotopy groups of $\mathrm{O} = \mathrm{colim}\ \mathrm{O}(n)$ are famously given by the Bott periodicity theorem:

$n \pmod 8$	0	1	2	3	4	5	6	7
$\pi_n\mathrm{O}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

Altogether, we find that $\pi_*(B\mathrm{SO})$ is $\mathbb{Z}/2$ when $* \equiv 1$ or $2 \bmod 8$, $\mathbb{Z}$ when $* \equiv 0 \bmod 4$, and 0 otherwise. Tensoring with $\mathbb{Q}$ kills the torsion groups, so we find that $\pi_*(B\mathrm{SO}) \otimes \mathbb{Q}$ is rank 1 in dimensions $4n > 0$. □

Exercise 1.2. Prove that direct sums of principal SO-bundles gives $B\mathrm{SO}$ the structure of an H-space.

Remark 1.3. Last time, we had following question: why is $H_*(B\mathrm{SO};\mathbb{Q})$ a ring instead of just a sequence of groups? Well, this is a consequence of $B\mathrm{SO}$ being an H-space. Given two topological spaces X and Y, taking products of singular simplices gives us a map $H_m(X;R)\otimes H_n(Y;R)\to H_{m+n}(X\times Y;R)$. If $X=Y$ is an H-space, then we can use the multiplication map $\mu : X\times X\to X$ to obtain a map

$$H_m(X;R)\otimes H_n(X;R)\to H_{m+n}(X\times X;R)\xrightarrow{\mu_*} H_{m+n}(X;R).$$

This is sometimes called the *Pontryagin product*. You can check that this turns $H_*(X;R)$ into a graded ring. Last time, we showed that $\pi_*(B\mathrm{SO})\otimes\mathbb{Q}$ and $H_*(B\mathrm{SO};\mathbb{Q})$ are isomorphic as sequences of abelian groups. To finish off the proof, we need to check that the Pontryagin product is compatible with the graded ring structure on $\pi_*(B\mathrm{SO})\otimes\mathbb{Q}$. I'll leave this as an unofficial exercise.

Note that I haven't told you where the graded ring structure on stable homotopy groups comes from — if you haven't seen this yet, don't worry about it for now. I'm saving it for when we dive a little deeper into spectra (a few weeks from now).

The generators p_i are typically chosen to be the *Pontryagin classes*, which belong in any basic toolkit of cohomology classes.

We're almost there. The last thing we need to do is explain why $H_*(B\mathrm{SO};\mathbb{Q})$ can be generated by $\{\mathbb{CP}^2,\mathbb{CP}^4,\ldots\}$ instead of $\{p_1,p_2,\ldots\}$.

Exercise 1.4. Building on last lecture, show that the signature of a $4n$-manifold induces a group homomorphism $\sigma : \Omega^{\mathrm{SO}}_{4n}\to\mathbb{Z}$. Prove that $\sigma(\mathbb{CP}^{2n})=1$, and deduce that $\{\mathbb{CP}^2,\mathbb{CP}^4,\ldots\}$ generate $\Omega^{\mathrm{SO}}_*\otimes\mathbb{Q}$.

2. The L-genus and Hirzebruch signature theorem

Last week, we saw that the signature of a 4-manifold is a cobordism invariant. The same style of proof can be used to show that the signature of a $4n$-manifold is a cobordism invariant. If we define the signature of an n-manifold to be 0 whenever $4\nmid n$, and using the Künneth formula to prove that $\sigma(M_1\times M_2)=\sigma(M_1)\cdot\sigma(M_2)$, we find that $\sigma:\Omega^{\mathrm{SO}}_*\to\mathbb{Z}$ is a *genus.*

We could write down the logarithm and characteristic series of this genus, but you would be left still wondering, "How did anybody come up with these power series? Why not take a different type of generating function?" Well, let's dive into the history. While working on cobordism theory, Thom discovered a relationship between the signature of 4-manifolds and their Pontryagin classes.

Definition 2.1. Given a manifold M, the k^{th} *Pontryagin class* is defined to be

$$p_k(M):=(-1)^k c_{2k}(TM\otimes\mathbb{C})\in H^{4k}(M;\mathbb{Z}),$$

where c_{2k} is the $2k^{\mathrm{th}}$ Chern class.

Theorem 2.2 (Thom). *Let M be a compact, oriented 4-manifold. Then* $\sigma(M) = \int_M \frac{p_1}{3} := \langle \frac{p_1(M)}{3}, [M] \rangle$.

Proof. Since σ and p_1 both determine linear maps $\Omega_4^{\mathrm{SO}} \otimes \mathbb{Q} \xrightarrow{\cong} \mathbb{Q}$, they must differ by a constant. It thus suffices to compute $\int_{\mathbb{CP}^2} p_1 = 3$, which you will do in the next exercise. ◻

Exercise 2.3. Prove that $p_k(\mathbb{CP}^n) := p_k(T\mathbb{CP}^n) = \binom{n+1}{k}$ for $1 \le k \le n/2$.

Hint: there is an isomorphism of vector bundles $T\mathbb{CP}^n \oplus \mathcal{O} \cong \mathcal{O}(1)^{\oplus n+1}$. The total Chern class of $\mathcal{O}(1)$ is $c(\mathcal{O}(1)) = 1 + x$, where $x \in H^2(\mathbb{CP}^2)$ is a hyperplane class. Use this to compute $c(T\mathbb{CP}^n) = c(\mathcal{O}(1))^{n+1}$, which will tell you $c_k(T\mathbb{CP}^n)$ for $1 \le k \le n$. Finally, use the identity

$$(1 - p_1 + p_2 - \cdots \pm p_n) = (1 - c_1 + c_2 - \cdots \pm c_n)(1 + c_1 + c_2 + \cdots + c_n)$$

to deduce a formula for the total Pontryagin class $p(T\mathbb{CP}^n)$.

Theorem 2.2 is great, because it allows us to compute a global analytic invariant σ in terms of an algebraic invariant p_1. In 1953, Hirzebruch was thinking about how to generalize this theorem to dimension $4n$ for any $n \ge 1$. He formulated the following conjecture:[1]

Conjecture 2.4 (Hirzebruch). *Let*

$$L(x) := \frac{x}{\tanh x} = \sum_{k \ge 0} \frac{2^{2k} B_{2k} x^{2k}}{(2k)!},$$

where B_n is the n^{th} Bernoulli number. Let $L_n(p_1, \ldots, p_n)$ be the degree $4n$ term of $L(x_1) \cdots L(x_n)$, where x_i is a variable of degree 2 and p_i is the i^{th} elementary symmetric polynomial in the variables $x_1^2, \ldots, x_n^2$. Then for any compact, oriented $4n$-manifold M, we have

$$\sigma(M) = \int_M L_n(p_1, \ldots, p_n).$$

Remark 2.5. This seems like a crazy definition of the polynomials L_n. Where does it come from? Hirzebruch was hoping for polynomials L_n in the Pontryagin classes such that $\sigma(M) = \int_M L_n(p_1, \ldots, p_n)$. It turns out that $L(x)$ is the unique power series that is even and such that the coefficient of x^n in $L(x)^{2n+1}$ is 1. The condition that $L(x)$ is even comes from the fact that Pontryagin classes are even functions of the plane class x. The requirement that the coefficient of x^n in $L(x)^{2n+1}$ be 1 comes from applying the splitting principle.

There's a little more to the story — $L(x)$ needs to be the power series associated to a *multiplicative sequence* of polynomials. We won't have time to really explore this properly in class, but multiplicative sequences and the Hirzebruch signature theorem

[1] For Hirzebruch's explanation of how one might arrive at such a conjecture, see [Hir71, §2].

really showcase the interplay between geometry, topology, and analysis that we've been talking about. Let me know if you're interested in reading more about it and need help finding references!

Theorem 2.6 (Hirzebruch signature theorem)**.** *Conjecture 2.4 is true.*

Proof. We'll skip the mathematical details and just mention the historical drama. Hirzebruch was able to show that $M \mapsto \int_M L_n(p_1, \ldots, p_n)$ defined a ring homomorphism $\Omega_*^{\mathrm{SO}} \otimes \mathbb{Q} \to \mathbb{Q}$. Moreover, he could show that this ring homomorphism agreed with the ring homomorphism $\sigma : \Omega_*^{\mathrm{SO}} \otimes \mathbb{Q} \to \mathbb{Q}$ on the manifolds $\mathbb{CP}^{2n}$.

Shortly thereafter (on June 2, 1953 to be precise), fortune struck. Hirzebruch went to the library, pulled the latest edition of *Comptes Rendus* off the shelf, and saw Thom's new theorem: $\Omega_*^{\mathrm{SO}} \otimes \mathbb{Q} \cong \mathbb{Q}[\mathbb{CP}^2, \mathbb{CP}^4, \ldots]$. □

Remark 2.7. The power series $L(x)$ is the characteristic series of the *L-genus* or *signature genus*. There are analogs and generalizations of the Hirzebruch signature theorem, which relate genus values to polynomials coming from characteristic series. We'll see more of this later.

Exercise 2.8. Compute the logarithm of the L-genus. What does this tell you about the L-genus of $\mathbb{CP}^n$?

3. The Weierstrass ℘-function

The Hirzebruch signature theorem is a nice place to end our discussion of oriented cobordism. We'll soon come back to cobordism and see some a more general type of signature theorem. But we first need to take a detour through the world of modular forms.

Recall that an *elliptic function* is a doubly periodic, meromorphic function. These were discovered by inverting the three basic elliptic integrals. Today, I'll introduce a famous and remarkable elliptic function.

Definition 3.1. Let $\omega_1, \omega_2 \in \mathbb{C}$ be $\mathbb{R}$-linearly independent. Let Λ be the lattice generated by ω_1 and ω_2. The *Weierstrass ℘-function*[2] is the $\mathbb{C}$-valued function

$$\wp(z, \omega_1, \omega_2) := \wp(z, \Lambda) := \frac{1}{z^2} + \sum_{\lambda \in \Lambda - \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right).$$

If Λ is apparent from context, one often writes $\wp(z)$.

Remark 3.2. Just as the Jacobi elliptic functions arise from inverting elliptic integrals, one can discover $\wp(z)$ as the extension of $u^{-1}(z)$ to all of $\mathbb{C}$, where

$$u(z) = -\int_z^\infty \frac{\mathrm{d}t}{\sqrt{4t^3 - at - b}}$$

[2]The LaTeX for ℘ is `\wp`.

for some $a, b \in \mathbb{C}$ with $a^3 - 27b^2 \neq 0$.

We have a few things to prove about $\wp(z)$. The first is that its definition actually converges at $z \in \mathbb{C} - \Lambda$.

Lemma 3.3. *The series*

$$\frac{1}{z^2} + \sum_{\lambda \in \Lambda - \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$$

converges absolutely on $\mathbb{C} - \Lambda$ and uniformly on compact sets.

Exercise 3.4. Prove Lemma 3.3.

Hint: show that for any $r > 0$, the series converges uniformly on $\{|z| \leq r\}$ by splitting the sum into a finite sum with poles and an infinite sum. A finite sum with poles is meromorphic and converges uniformly absolutely. You'll have to show more directly that the infinite sum converges uniformly and absolutely.

Now we can show that $\wp(z)$ is doubly periodic with respect to ω_1 and ω_2.

Lemma 3.5. *We have $\wp(z) = \wp(z + \omega_1) = \wp(z + \omega_2)$ for all $z \in \mathbb{C}$.*

Proof. Since the summation definition converges absolutely, we can reorder the terms in the sum. Now pick ω_1 and ω_2 that generate Λ. Can you see from the summation why $\wp(z + \omega_1) = \wp(z + \omega_2) = \wp(z)$? □

By construction, we can see that $\wp(z)$ has a double pole at λ for each $\lambda \in \Lambda$. In fact, $\wp(z)$ is meromorphic.

Lemma 3.6. *The function $\wp(z)$ is meromorphic with derivative*

$$\wp'(z) = -2 \sum_{\lambda \in \Lambda} \frac{1}{(z-\lambda)^3}.$$

Proof. The derivative just comes from the power rule. To see that $\wp(z)$ is meromorphic, pick $r > 0$ such that $\{z \in \mathbb{C} : |z| \leq r\}$ contains a fundamental domain of $\wp(z)$. We have seen that the summation for $\wp(z)$ is uniformly convergent on this set, so $\wp(z)$ is uniformly convergent on its fundamental domain. By double periodicity, we find that $\wp(z)$ is uniformly convergent (and hence meromorphic) on all of $\mathbb{C}$. □

Next time, we'll learn some more about $\wp$. This will serve as our introduction to modular forms in general.

Next time: More $\wp$ and modular forms.

Daily exercises: I decided to stop collecting the exercises here. If you really want me to put them at the end of the notes like before, let me know!

References

[Hir71] F. Hirzebruch. "The signature theorem: reminiscences and recreation". In: *Prospects in mathematics (Proc. Sympos., Princeton Univ., Princeton, N.J., 1970)*. Vol. No. 70. Ann. of Math. Studies. Princeton Univ. Press, Princeton, N.J., 1971, pp 3–31. URL: https://www.maths.ed.ac.uk/~v1ranick/papers/hirzrem.pdf.

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