LECTURE 8: THE WEIERSTRASS $\wp$-FUNCTION AND FRIENDS

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Last time, we met the Weierstrass $\wp$-function

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda - \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right),$$

which is the sort of thing you end up with if you try to manually construct a doubly periodic meromorphic function.

Remark 0.1. Just as the Jacobi elliptic functions arise from inverting elliptic integrals, one can discover $\wp(z)$ as the extension of $u^{-1}(z)$ to all of $\mathbb{C}$, where

$$u(z) = -\int_z^\infty \frac{\mathrm{d}t}{\sqrt{4t^3 - at - b}}$$

for some $a, b \in \mathbb{C}$ with $a^3 - 27b^2 \neq 0$.

1. The Weierstrass $\wp$-function

We left as an exercise that the summation notation for $\wp(z)$ satisfies nice convergence properties. One of these is that the sum converges absolutely (away from the poles), which means that we can reorder the summation. The ability to reorder the summation is how we showed that $\wp$ is indeed doubly periodic.

By construction, we can see that $\wp(z)$ has a double pole at λ for each $\lambda \in \Lambda$. In fact, $\wp(z)$ is meromorphic.

Lemma 1.1. *The function $\wp(z)$ is meromorphic with derivative*

$$\wp'(z) = -2 \sum_{\lambda \in \Lambda} \frac{1}{(z-\lambda)^3}.$$

Proof. The derivative just comes from the power rule. To see that $\wp(z)$ is meromorphic, pick $r > 0$ such that $\{z \in \mathbb{C} : |z| \leq r\}$ contains a fundamental domain of $\wp(z)$. We left as an exercise that the summation for $\wp(z)$ is uniformly convergent on this set, so $\wp(z)$ is uniformly convergent on its fundamental domain. By double periodicity, we find that $\wp(z)$ is uniformly convergent (and hence meromorphic) on all of $\mathbb{C}$. □

So now we've got a shiny, new elliptic function. What should we do with it? The first thing you might try is expressing $\wp(z)$ as a Laurent series to get a better sense of its behavior.

Lemma 1.2. *Let* $r = \min\{|\lambda| : \lambda \in \Lambda - \{0\}\}$. *The Laurent series of* $\wp(z)$ *for* $|z| < r$ *is given by*

$$\wp(z) = \frac{1}{z^2} + \sum_{n\geq 1}(2n+1)\left(\sum_{\lambda\in\Lambda-\{0\}} \lambda^{-(2n+2)}\right) z^{2n}.$$

Proof. Since the summation formula for $\wp(z)$ converges absolutely, we can rearrange the summands. Recall that

$$-1 + \frac{1}{(x-1)^2} = -1 + \frac{1}{(1-x)^2} = \sum_{n\geq 1}(n+1)x^n$$

for $|x| < 1$. For each $\lambda \in \Lambda - \{0\}$, the definition of r implies that $|z/\lambda| < 1$ for $|z| < r$. Using this fact, we get

$$\begin{aligned}
\wp(z) &= \frac{1}{z^2} + \sum_{\lambda\in\Lambda-\{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right) \\
&= \frac{1}{z^2} + \sum_{\lambda\in\Lambda-\{0\}} \left(\frac{1}{(\frac{z}{\lambda}-1)^2}\frac{1}{\lambda^2} - \frac{1}{\lambda^2}\right) \\
&= \frac{1}{z^2} + \sum_{\lambda\in\Lambda-\{0\}} \frac{1}{\lambda^2}\left(\sum_{n\geq 1}(n+1)\left(\frac{z}{\lambda}\right)^n\right) \\
&= \frac{1}{z^2} + \sum_{n\geq 1}(n+1)\left(\sum_{\lambda\in\Lambda-\{0\}} \lambda^{-(n+2)}\right) z^n.
\end{aligned}$$

It remains to show that the odd powers vanish (which you should expect, since our usual definition of $\wp(z)$ is clearly an even function). If n is odd, then $\lambda^{-(n+2)} = -(-\lambda)^{-(n+2)}$. Since $\lambda, -\lambda \in \Lambda$, it follows that $\sum_{\lambda\in\Lambda-\{0\}} \lambda^{-(n+2)} = 0$ for every odd n. □

Psychologically, this feels a little more familiar than our original formulation of $\wp(z)$ — except for the coefficients $G_{2n+2} := \sum_{\lambda\in\Lambda-\{0\}} \lambda^{-(2n+2)}$. What are we supposed to make of these? We'll get to this in a second. But first, an exercise:

Exercise 1.3. Let $g_2 := 60G_4$ and $g_3 := 140G_6$. Prove that $\wp(z)$ satisfies the differential equation

$$\wp'(z)^2 = 4\wp(z) - g_2\wp(z) - g_3.$$

Hint: form a linear combination of powers of $\wp$ and $\wp'$ in a way that eliminates the pole at $z = 0$. Such a linear combination preserves the periodicity, so eliminating the pole at $z = 0$ removes all poles and yields an entire function. Now think about why (and how) we can apply Liouville's theorem in this situation.

1.1. **Elliptic curves are not ellipses.** Here's an amazing fact about g_2 and g_3. Since $\wp(z)$ is periodic over the lattice Λ, it gives us a well-defined function on $\mathbb{C}/\Lambda$. Topologically, $\mathbb{C}/\Lambda$ is a torus. But in fact, $\mathbb{C}/\Lambda$ is the complex projective curve cut out by the equation

$$y^2z = 4x^3 - g_2xz^2 - g_3z^3.$$

Note that this is the homogenization of the differential equation in the previous exercise. So not only does the lattice Λ determine the *elliptic curve* $\mathbb{C}/\Lambda$, but we can explicitly recover its defining equation from Λ. Let's actually prove this claim.

Lemma 1.4. *Let $\Lambda \subset \mathbb{C}$ be a lattice with Weierstrass function $\wp(z)$. Then there is a biholomorphism from $\mathbb{C}/\Lambda$ to the complex projective curve cut out by $y^2z = 4x^3 - g_2xz^2 - g_3z^3$.*

Proof. Let $E := \{[x : y : z] \in \mathbb{CP}^2 : y^2z = 4x^3 - g_2xz^2 - g_3z^3\}$. Define $\varphi : \mathbb{C} \to E$ by

$$\varphi(z) = \begin{cases} [\wp(z) : \wp'(z) : 1] & z \in \mathbb{C} - \Lambda, \\ [0 : 1 : 0] & z \in \Lambda. \end{cases}$$

Since $\wp$ and $\wp'$ are periodic with respect to Λ, we get a well-defined map $\bar{\varphi} : \mathbb{C}/\Lambda \to E$. Moreover, both $\wp$ and $\wp'$ are holomorphic away from Λ, so $\bar{\varphi}$ is holomorphic away from the point that we've collapsed Λ to. Even better, we've compactified our source and target, so $\bar{\varphi}$ no longer has a pole on the image of Λ — $\bar{\varphi}$ is holomorphic on all of $\mathbb{C}/\Lambda$.

It remains to show that $\bar{\varphi}$ is bijective, since a bijective holomorphic function is automatically biholomorphic (one of the many miracles of complex analysis). Surjectivity comes from a formal argument: φ is a non-constant holomorphic map, so it is an open map. It follows that $\bar{\varphi}$ is an open map between compact spaces, so $\bar{\varphi}$ is surjective.

To see injectivity, we first note that $\wp(z) : \mathbb{C}/\Lambda \to \mathbb{C} \cup \{\infty\}$ is two-to-one (or has *degree 2*). We even know how each pair of preimages looks. Since $\wp(z)$ is an even function, $\wp^{-1}(\wp((z)) = \{\pm z\}$ (except at $z = -z$, where we have the double pole $\wp(z) = \infty$). To rephrase, $\wp(z) = \wp(z')$ if and only if $z = \pm z'$.

So to see that $\bar{\varphi}$ is injective, we just need to show that $\wp'(z) \neq \wp'(-z)$ for any $z \neq -z$. Since $\wp'(z)$ is an odd function, $\wp'(z) = \wp'(-z)$ implies $\wp'(z) = 0$. But $\wp'(z)$ implies that $\wp$ has a double point at z, and the double points of $\wp$ only occur when $z = -z$. In particular, $\wp'(z) = \wp'(-z)$ implies that $\wp$ has a single preimage at $\wp(z)$, so $\bar{\varphi}$ is injective. □

This is how elliptic curves were discovered historically, which explains the confusing terminology: elliptic curves are tori, not ellipses. We'll talk more about elliptic curves next time.

2. Eisenstein series

To get a handle on G_{2n+2}, let's express these as a sum over the integer lattice, with a complex variable accounting for the shift from the integer lattice to Λ. Up to scaling and rotation, every lattice is generated by 1 and $\tau \in \mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. This leads us to the notion of Eisenstein series.

Definition 2.1. The *Eisenstein series of weight* $2k$ is the function

$$G_{2k}(\tau) := \sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}} (m+n\tau)^{-2k}$$

on the upper half-plane $\mathbb{H}$.

Remark 2.2. Before studying any properties of $G_{2k}(\tau)$, one should check that this series converges absolutely for $k \geq 2$. The argument is more or less the same as the one used to check that our original summation formula for $\wp(z)$ converges absolutely, which was left to you as an exercise.

Alright, on to the miraculous properties of $G_{2k}(\tau)$.

Lemma 2.3 (Translation invariance)**.** *If $k \geq 2$, then $G_{2k}(\tau+1) = G_{2k}(\tau)$.*

Proof. Since the defining sum for $G_{2k}(\tau)$ converges absolutely, we can rearrange summands, compute $m + n(\tau+1) = m + n + n\tau$, and check that the function

$$\begin{aligned} \mathbb{Z}^2 - \{(0,0)\} &\to \mathbb{Z}^2 - \{(0,0)\} \\ (m,n) &\mapsto (m+n,n) \end{aligned}$$

is bijective. The inverse is given by $(a,b) \mapsto (a-b,b)$. □

Lemma 2.4 (Inversion)**.** *If $k \geq 2$, then $G_{2k}(-1/\tau) = \tau^{2k}G_{2k}(\tau)$.*

Proof. First, we need to check that $-1/\tau \in \mathbb{H}$. Indeed, if $b > 0$, then $-1/(a+ib) = -(a-ib)/(a^2+b^2)$, which has positive imaginary part. Now

$$\begin{aligned} G_{2k}(-1/\tau) &= \sum_{m,n}(m-n/\tau)^{-2k} \\ &= \sum_{m,n}\tau^{2k}(m\tau-n)^{-2k} \\ &= \tau^{2k}\sum_{n,-m}(n\tau+m)^{2k} \\ &= \tau^{2k}G_{2k}(\tau). \end{aligned}$$

□

Exercise 2.5 (Automorphy)**.** Let $k \geq 2$. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. Show that γ acts on $\mathbb{H}$ by $\tau \mapsto \frac{a\tau+b}{c\tau+d}$. Then prove that $G_{2k}(\gamma\cdot\tau) = (c\tau+d)^{2k}\cdot G_{2k}(\tau)$.

Exercise 2.6 (Bounded growth)**.** Prove that if $k \geq 2$, then there exists some $A, B > 0$ such that $|G_{2k}(\tau)| < A$ for all τ with $\mathrm{Im}(\tau) > B$.

2.1. **Modular forms.** The Eisenstein series are our first examples of *modular forms*. We just saw that they G_{2k} satisfies three interesting symmetry relations and one condition on its growth. These seem like very strong properties for a function to have, and yet G_{2k} give us a whole family of non-trivial functions satisfying these properties. Anytime you have a family of objects satisfying surprisingly strong conditions, you should make a definition and look for more examples.

Definition 2.7. A *modular form of weight k* is a complex function $f : \mathbb{H} \to \mathbb{C}$ satisfying three conditions:

(i) f is holomorphic on $\mathbb{H}$.

(ii) f satisfies *weight k automorphy*: for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{SL}_2(\mathbb{Z})$, we have $f(\gamma \cdot \tau) = (c\tau + d)^k f(\tau)$.

(iii) There exist $A, B > 0$ such that $|f(\tau)| < A$ for all τ with $\mathrm{Im}(\tau) > B$.

Exercise 2.8. Show that every modular form of weight k satisfies translation invariance ($f(\tau + 1) = f(\tau)$) and inversion ($f(-1/\tau) = \tau^k f(\tau)$).

Next time, we'll see lots more examples of modular forms. For now, see if you can use G_{2k} to cook up new examples!

Daily exercises: I decided to stop collecting the exercises here. If you really want me to put them at the end of the notes like before, let me know!

Next time: more modular forms, then elliptic curves.

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