LECTURE 9: MODULAR FORMS

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Last time, we gave the following Laurent series expansion of the Weierstrass $\wp$-function:

$$\wp(z,\Lambda) = \frac{1}{z^2} + \sum_{n\geq 1}(n+1)\left(\sum_{\lambda\in\Lambda-\{0\}} \lambda^{-(2n+2)}z^{2n}\right).$$

We then defined the *Eisenstein series of weight* $2k$ to be

$$G_{2k}(\Lambda) := \sum_{\lambda\in\Lambda-\{0\}} \lambda^{-2k}$$

for $k\geq 2$. We then learned that if we represent our lattice Λ as depending on a single parameter of the upper half-plane, G_{2k} becomes truly remarkable:

(i) Translation invariance: $G_{2k}(\tau+1) = G_{2k}(\tau)$.

(ii) Inversion: $G_{2k}(-1/\tau) = \tau^{2k}G(\tau)$.

(iii) Automorphy: $G_{2k}(\gamma\cdot\tau) = (c\tau+d)^{2k}G(\tau)$ for any $\gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

(iv) Bounded growth: there exist $A, B > 0$ such that $|G_{2k}(\tau)| < A$ for each τ such that $\mathrm{Im}(\tau) > B$.

Today, we're going to talk about functions satisfying these properties. But first, I want to officially write down an exercise for you. The point of this exercise is to help you get a sense of how lossy (or not) our restriction from $G_{2k}(\Lambda)$ to $G_{2k}(\tau)$ might be.

Exercise 0.1. Let Λ be a lattice. Let Λ' be a the lattice given by rotating Λ by angle θ and stretching by $r > 0$. What is the relationship between $G_{2k}(\Lambda)$ and $G_{2k}(\Lambda')$?

1. Modular forms

The Eisenstein series are our first examples of *modular forms*. We just saw that they G_{2k} satisfies three interesting symmetry relations and one condition on its growth. These seem like very strong properties for a function to have, and yet G_{2k} give us a whole family of non-trivial functions satisfying these properties. Anytime you have a family of objects satisfying surprisingly strong conditions, you should make a definition and look for more examples.

Definition 1.1. A *modular form of weight* k is a complex function $f : \mathbb{H} \to \mathbb{C}$ satisfying three conditions:

(i) f is holomorphic on $\mathbb{H}$.

(ii) f satisfies *weight* k *automorphy*: for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{SL}_2(\mathbb{Z})$, we have $f(\gamma \cdot \tau) = (c\tau + d)^k f(\tau)$.

(iii) There exist $A, B > 0$ such that $|f(\tau)| < A$ for all τ with $\mathrm{Im}(\tau) > B$.

Exercise 1.2. Show that every modular form of weight k satisfies translation invariance ($f(\tau + 1) = f(\tau)$) and inversion ($f(-1/\tau) = \tau^k f(\tau)$).

We've made our definition of modular forms based on the examples we have at hand. Now it's time to look for more examples. Your first instinct should always be to check if your new definition admits a group or even ring structure.

Lemma 1.3. *If f and g are modular forms of weight k, then $f + g$ is a modular form of weight k.*

Proof. There are three things to check:

(i) Holomorphicity. The sum of two holomorphic functions is again holomorphic, so we're good here.

(ii) Automorphy. If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, then

$$\begin{aligned} (f+g)(\gamma \cdot \tau) &= f(\gamma \cdot \tau) + g(\gamma \cdot \tau) \\ &= (c\tau + d)^k f(\tau) + (c\tau + d)^k g(\tau) \\ &= (c\tau + d)^k (f + g)(\tau). \end{aligned}$$

(iii) Bounded growth. If we have $A, B, C, D > 0$ such that $|f(\tau)| < A$ for all $\mathrm{Im}(\tau) > B$ and $|g(\tau)| < C$ for all $\mathrm{Im}(\tau) > D$, then $|f(\tau) + g(\tau)| < A + C$ for all $\mathrm{Im}(\tau) > \max\{B, D\}$ by the triangle inequality. □

Lemma 1.4. *If f is a modular form of weight k, then so is $-f$.*

Proof. Negation doesn't affect the holomorphicity of a function. Automorphy follows by multiplying the automorphy equation for f by -1. Bounded growth follows from $|-f| = |f|$. □

Corollary 1.5. *The zero function is a modular form of weight k for all k. Moreover, the set of all modular forms of a fixed weight k forms an abelian group under addition.*

Proof. All of the properties of modular forms hold for the zero function, and the previous two lemmas were checking the group operation and negation. Associativity and commutativity follow from the fact that addition of functions is associative and commutative. □

Remark 1.6. You can check that if f is a modular form of weight k, then $c \cdot f$ is a modular form of weight k for any $c \in \mathbb{C}$. It follows that the abelian group of weight k modular forms is even a complex vector space.

Alright, what about multiplying modular forms?

Lemma 1.7. *Let f_1 and f_2 be modular forms of weights k_1 and k_2, respectively. Then $f_1 \cdot f_2$ is a modular form of weight $k_1 + k_2$.*

Proof. Holomorphicity is preserved under multiplication. For bounded growth, the check boils down to noting that $|f_1 \cdot f_2| = |f_1| \cdot |f_2|$. It remains to check automorphy:

$$\begin{aligned} f_1 \cdot f_2(\gamma \cdot \tau) &= f_1(\gamma \cdot \tau) \cdot f_2(\gamma \cdot \tau) \\ &= (c\tau + d)^{k_1} f_1(\tau) \cdot (c\tau + d)^{k_2} f_2(\tau) \\ &= (c\tau + d)^{k_1+k_2} f_1 \cdot f_2(\tau). \end{aligned}$$

□

Altogether, we've shown that modular forms fit together in a graded ring.

Corollary 1.8. *Let MF_k denote the abelian group of modular forms of weight k. Then $\mathrm{MF}_* = \bigoplus_{k \geq 0} \mathrm{MF}_k$ is a graded ring.*

1.1. **Fundamental domain.** Thanks to automorphy, we really only need to understand the behavior of a modular form on a fundamental domain of the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\bar{\mathbb{H}} := \mathbb{H} \cup \{z : \mathrm{Im}(z) = 0\} \cup \{\infty\}$. I'll draw a picture of the fundamental domain in class, but it's an excellent exercise to work it out for yourself.

Exercise 1.9. Prove that the fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\bar{\mathbb{H}}$ is $\{z \in \mathbb{H} : |z| \geq 01 \text{ and } -1/2 \leq \mathrm{Re}(z) \leq 0\} \cup \{z \in \mathbb{H} : |z| > 1 \text{ and } 0 < \mathrm{Re}(z) < 1/2\}$.

Hint: start by showing that $\mathrm{SL}_2(\mathbb{Z})$ is generated by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

1.2. **Examples.** Let's see a couple other modular forms.

Example 1.10 (Modular discriminant)**.** The most famous is the *modular discriminant*

$$\Delta(\tau) := g_2^3(\tau) - 27 g_3^2(\tau),$$

where $g_2 := 60G_4$ and $g_3 := 140G_6$. Since MF_* is a graded ring, we find that Δ is a modular form of weight 12. If you're curious about where Δ comes from, try taking the discriminant of $4x^3 - g_2 x - g_3$ (the right hand side of the ODE for $\wp$ from last time).

Example 1.11 (Theta functions of even unimodular lattices)**.** A lattice $\Lambda \subset \mathbb{R}^n$ is *unimodular* if it is generated by n vectors who form the columns of a matrix of determinant 1. A unimodular lattice Λ is called *even* if its generating vectors $v_1, \ldots, v_n$ all satisfy $\langle v_i, v_i \rangle \in 2\mathbb{Z}$.

Here's a particular construction of even unimodular lattices that will be useful in today's exercises. Let $n \geq 1$. Let $\Lambda \subset \mathbb{R}^{8n}$ be the lattice consisting of vectors λ such that $2\lambda \in \mathbb{Z}^{8n}$, with all components of 2λ even or all components odd, and such that the sum of the components of λ is an even integer. We denote this lattice by L_{8n}.

Given a lattice $\Lambda \subset \mathbb{R}^n$, define the *theta function*

$$\vartheta_\Lambda(\tau) := \sum_{\lambda \in \Lambda} e^{\pi i \langle \lambda, \lambda \rangle \tau},$$

where $\tau \in \mathbb{H}$.[1] If $\Lambda \subset \mathbb{R}^n$ is even and unimodular, then ϑ is a modular form of weight $k/2$:

(i) To see that $\vartheta_\Lambda(\tau)$ is holomorphic, note that $\tau \in \mathbb{H}$ implies that $\mathrm{Re}(i\tau) < 0$. In particular, the real part of each summand of $\vartheta_\Lambda(\tau)$ is negative. Now for any radius $r > 0$, there are finitely many lattice points within radius r of the origin. We can thus apply the same techniques we used in proving that $\wp$ is absolutely convergent and converges uniformly on compact sets.

(ii) Since $\mathrm{SL}_2(\mathbb{Z})$ is generated by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, it suffices to check automorphy for under the action of these two matrices. In other words, we need to check that $\vartheta_\Lambda(-1/\tau) = \tau^{k/2}\vartheta_\Lambda(\tau)$ and $\vartheta_\Lambda(\tau+1) = \vartheta_\Lambda(\tau)$.

The latter is easy to check: our assumption that Λ is *even* implies that $\langle \lambda, \lambda \rangle$ is an even integer for all $\lambda \in \Lambda$, so $e^{\pi i \langle \lambda, \lambda \rangle} = 1$. This gives us translation invariance.

Inversion is slightly trickier, and we'll have to skip it for time's sake. Here's the rough idea; read more about it if it sounds interesting to you! Our assumption that Λ is unimodular implies that $\Lambda = \Lambda^*$. Now the *Poisson summation formula* relates $\sum_\Lambda f(\lambda) = \frac{1}{\mu(\mathbb{R}^n/\Lambda)} \sum_{\Lambda^*} \hat{f}(\lambda')$, where $f(\lambda) = e^{\pi i \langle \lambda, \lambda \rangle \tau}$ and $\hat{f}$ is the Fourier transform. Applying this formula carefully results in the desired inversion formula.

(iii) Bounded growth comes from our observation in (i). Writing $\tau = x + iy$ with $y > 0$, we have

$$\begin{aligned} |\vartheta_\Lambda(\tau)| &\leq \sum_{\lambda \in \Lambda} |e^{\pi i \langle \lambda, \lambda \rangle \tau}| \\ &= \sum_{\lambda \in \Lambda} |e^{-\pi \langle \lambda, \lambda \rangle y}| \cdot |e^{\pi i \langle \lambda, \lambda \rangle x}| \\ &= \sum_{\lambda \in \Lambda} e^{-\pi \langle \lambda, \lambda \rangle y}. \end{aligned}$$

One can now directly check that this satisfies bounded growth.

[1] The LaTeX for ϑ is `\vartheta`.

Exercise 1.12. Show that L_{8n} is even and unimodular for each $n \geq 1$. Then show that the lattices $L_8 \times L_8$ and L_{16} are not similar. Deduce that the manifolds $\mathbb{R}^{16}/(L_8 \times L_8)$ and $\mathbb{R}^{16}/L_{16}$ are not isometric.

I could go on and on, but we're about to see that these examples are superfluous from one point of view.

2. There aren't that many modular forms

Although MF_* is a ring, its objects are analytic objects. In my personal experience, whenever you have a ring of analytic objects, that ring is usually infinitely generated or some other sort of nonsense. Not so for MF_*:

Theorem 2.1. *There is an isomorphism* $\mathrm{MF}_* \cong \mathbb{C}[G_4, G_6]$ *of* $\mathbb{C}$*-algebras.*

Proof. Two good places to see a proof are [Zag08, §2.1] or [Lan95, §2]. Here's the general idea. The $\mathbb{C}$-algebra structure on MF_* comes from viewing each MF_k as a $\mathbb{C}$-vector space. The proof now proceeds via some dimension counting and a check that G_4 and G_6 are algebraically independent. If we have time next class, we'll do this in more detail. □

Remark 2.2. You should now be saying, "Wait a minute, what about all the other Eisenstein series?" Well, they satisfy a pretty crazy recurrence relation. Let $d_k = (2k+3)k!G_{2k+4}$ for $k \geq 0$. Then

$$d_{n+2} = \frac{3n+6}{2n+9} \sum_{k=0}^{n} \binom{n}{k} d_k d_{n-k}$$

for all $n \geq 0$. We won't prove this, but you might find it to be a fun exercise.

In particular, once you know G_4 and G_6, the remaining Eisenstein series are given by a polynomial in G_4 and G_6. The coefficients of such a polynomial are rational — are they ever integers?

2.1. **You can't hear the shape of a drum.**

As an application of this theorem, let's come back to those theta functions of even unimodular lattices.

Exercise 2.3. Prove that $\vartheta_{L_8 \times L_8}(\tau) = \vartheta_{L_{16}}(\tau)$.

Exercise 2.4. Let $\Lambda, \Lambda_1, \Lambda_2 \subset \mathbb{R}^n$ be lattices.

(i) Show that the eigenbasis of the Laplacian $\nabla^2 = \sum_i \frac{\partial^2}{\partial x_i^2}$ on $\mathbb{R}^n/\Lambda$ consists of the functions $v \mapsto e^{2\pi i \langle \lambda^*, v \rangle}$ with eigenvalues $-(2\pi)^2 \langle \lambda^*, \lambda^* \rangle$, where $\lambda^* \in \Lambda^*$ is an element of the dual lattice.

(ii) Prove that $\mathbb{R}^n/\Lambda_1$ and $\mathbb{R}^n/\Lambda_2$ are *isospectral*[2] if and only if $\vartheta_{\Lambda_1} = \vartheta_{\Lambda_2}$. (Note that we are not assuming here that Λ_i are even or unimodular. The definition of the theta function still holds, although ϑ may not be a modular form.)

[2]Two manifolds are *isospectral* if their Laplacians have the same set of eigenvalues (counted with multiplicity).

Remark 2.5. Together, Exercises 1.12, 2.3, and 2.4 imply that $\mathbb{R}^{16}/(L_8 \times L_8)$ and $\mathbb{R}^{16}/L_{16}$ are isospectral but not isometric. In particular, these are two "drums" that do not "sound the same". This example was first observed by Milnor.

Daily exercises: I decided to stop collecting the exercises here. If you really want me to put them at the end of the notes like before, let me know!

Next time: elliptic curves.

References

[Lan95] Serge Lang. *Introduction to modular forms.* Vol. 222. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. With appendixes by D. Zagier and Walter Feit, Corrected reprint of the 1976 original. Springer-Verlag, Berlin, 1995, pp. x+261.

[Zag08] Don Zagier. "Elliptic modular forms and their applications". In: *The 1-2-3 of modular forms.* Universitext. Springer, Berlin, 2008, pp. 1–103. URL: https://doi.org/10.1007/978-3-540-74119-0_1.

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